Theory Working Group

Development and Analysis of Algorithms for problems traditionally handled via SMC.

SMC alg $\longrightarrow$ MCMC

Outline 1 Oct, 2008

- Perfect Simulation
  - Basic Acceptance/Rejection
- Randomness Recycler
- Examples
  - Hard Core Gas Model
  - Ising Model
- Relation to SST
- Initial Problems?

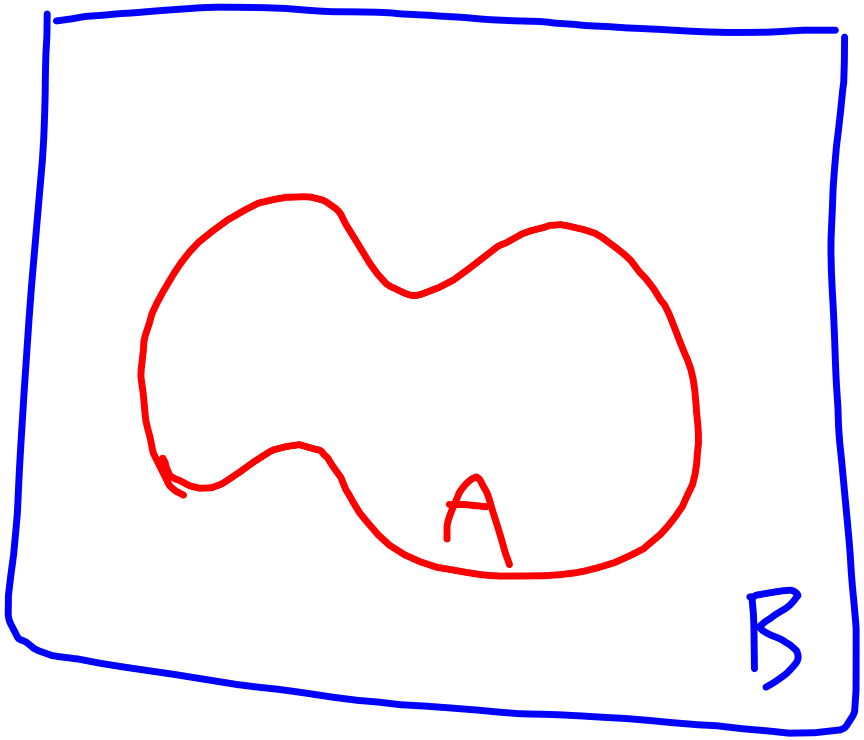
Perfect Simulation

Two salient features

1) Draws samples exactly from π.
2) Running time is a random variable (usually w/ exponential tails).

Basic Acceptance/Rejection

- Want uniform over A
- Can draw uniform over $B: A \subseteq B$



A/R

1) Repeat

2) Draw $X \leftarrow \text{Unif}(B)$

3) Until $X \in A$.

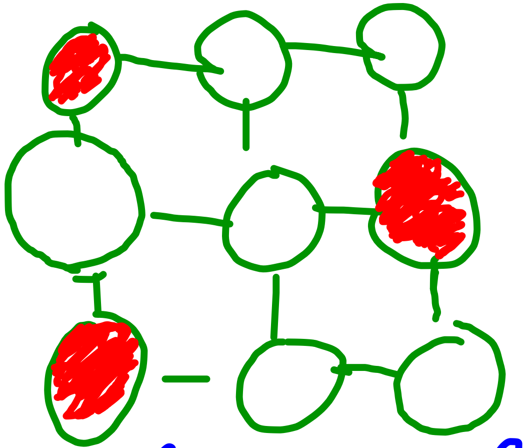
\# of times through loop $\sim \text{Geo}\left(\frac{\mu(A)}{\mu(B)}\right)$

random time w/ exponential tails

$$E[T] = \frac{\mu(B)}{\mu(A)}$$

Problem: $\frac{\mu(A)}{\mu(B)}$ usually decreases exponentially in size of problem.

Example: Hard Core Gas model



$$w(x) = \lambda^{\#x}$$

$$\pi(x) = \frac{w(x)}{Z_\lambda}$$

No 2 adjacent nodes in set

A/R for ind. sets aka Hard Core Gas...

Graph: $G=(V,E)$

$Z_\lambda \le (1+\lambda)^{|V|}$

v in set: $X(v)=1$

v out of set: $X(v)=0$

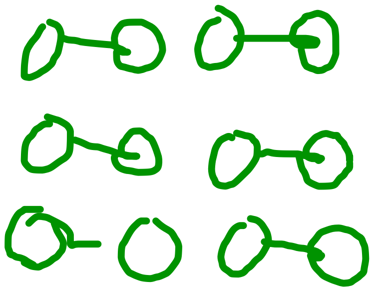
$\Omega = \{0,1\}^{|V|}$

AR ALG

1) Repeat
2) For all $v \in V$
3) Draw $x(v)$: $\mathbb{P}(x(v)=1) = \frac{\lambda}{1+\lambda}$
4) Until $\forall \{i,j\} \in E$ $x(i)x(j) = 0$.

Chance of no conflicts in lattice $\leq$

$$\left(\frac{2\lambda+1}{1+\lambda}\right)^{|V|/2}$$



A solution...

Randomness Recycler

Main Idea: Don't throw away sample completely when you reject

RR ALG ingredients

(a) State space Ex: $\{0,1\}^{|V|} = (\mathcal{X}, \mathcal{F})$

(b) Dual state space $(\mathcal{X}^*, \mathcal{F}^*)$

(c) Kernel Λ from $(\mathcal{X}^*, \mathcal{F}^*) \to (\mathcal{X}, \mathcal{F})$

(d) π target dist. on $(\mathcal{X}, \mathcal{F})$

(e) special dual state $x^*_\pi \in \mathcal{X}^*$

$$\Lambda(x^*_\pi, \cdot) = \pi(\cdot)$$

(f) special dual state x^*_0:

$\Lambda(x^*_0, \cdot)$ is easy to sample from

(g) bivariate Markov kernel on

$\mathcal{X}^* \times \mathcal{X}$

want to maintain

$$\mathcal{L}(X_t \mid X_0^* = x_0^*, \dots X_t^* = x_t^*) = \Pi(x_t^*, \cdot)$$

RR algorithm

1) Start X_0^* at x_0^*

2) Draw $X_0 \leftarrow \Delta(x_0^*, \cdot)$

3) Repeat

4) Take steps in Markov chain

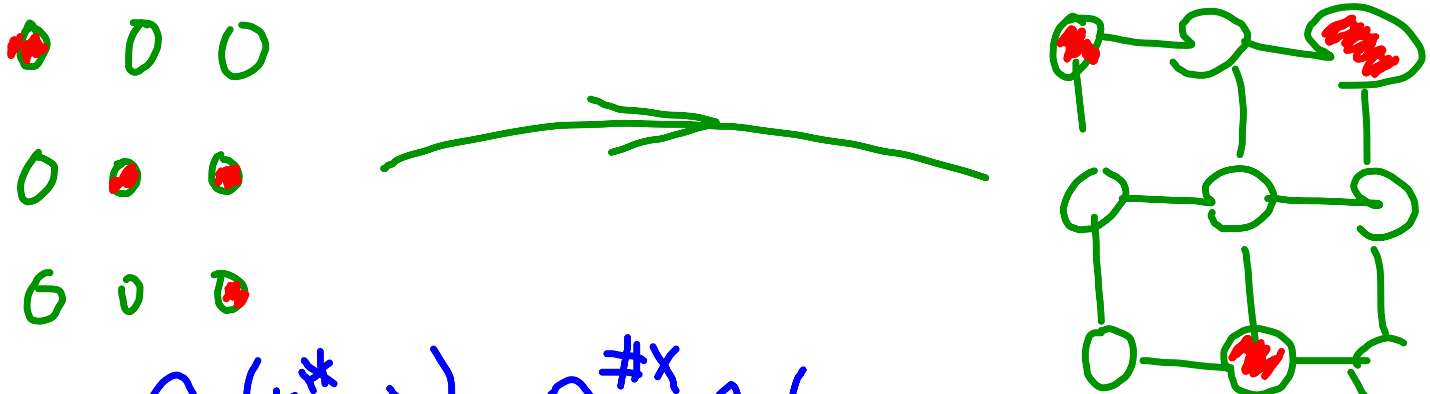
5) Until $X_t^* = X_\pi^*$

6) Output X_t.

Ex: Hard Core Gas Model

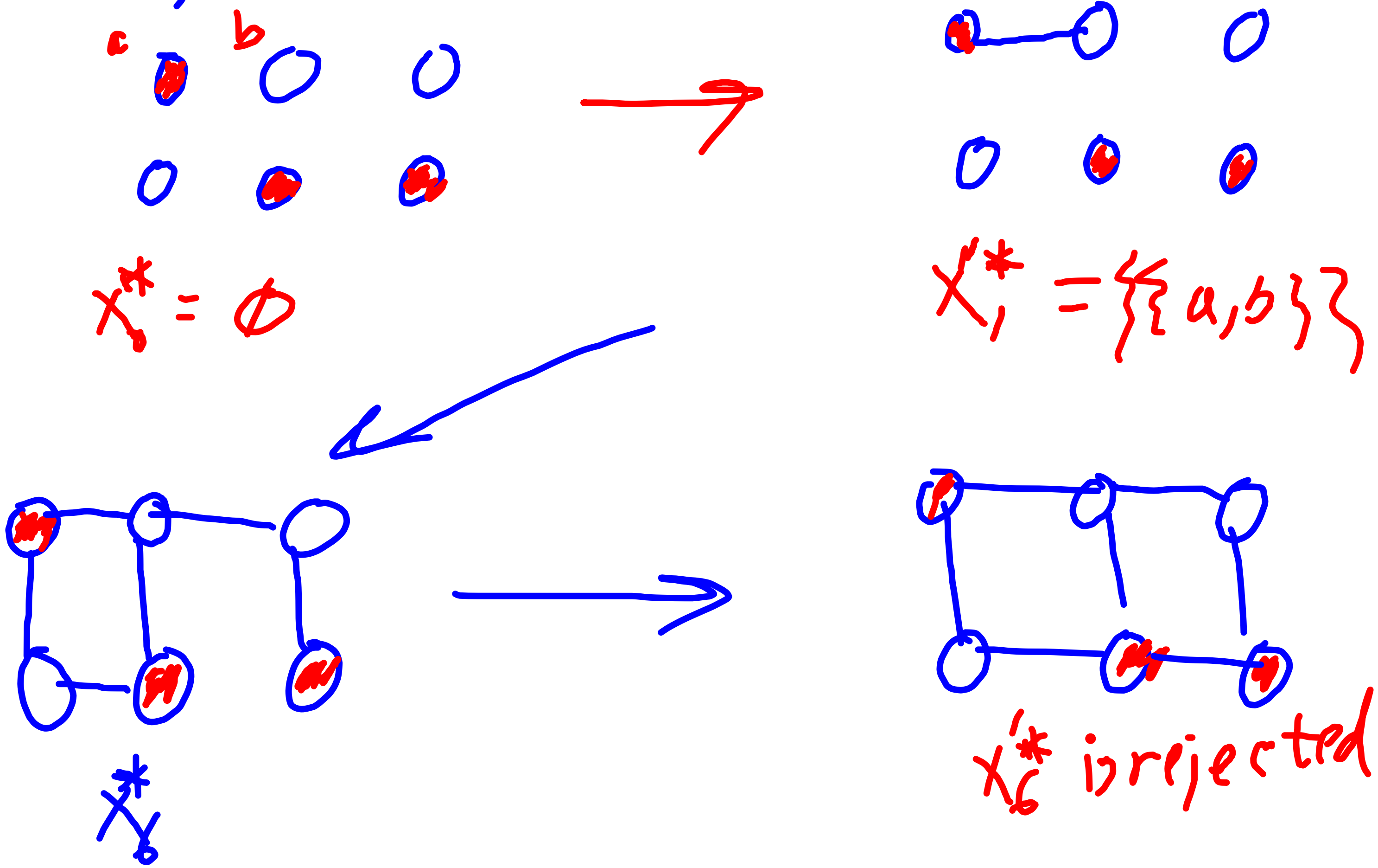
$$\mathcal{X}^* = 2^E$$

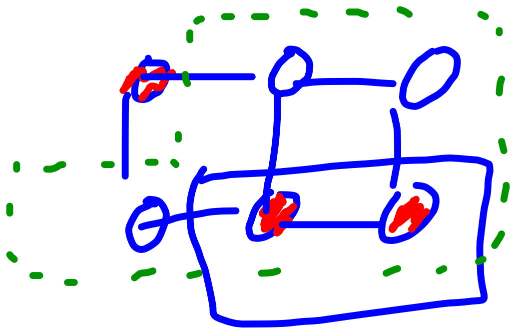
$$X_\pi^* = E, \quad X_0^* = \phi$$



$$\Delta(X_t^*, x) = \frac{\lambda^{\#x}\, \mathbb{1}(\forall e \in X_t^* : x(i)\,x(j) = 0)}{Z_{\lambda, X_t^*}}$$

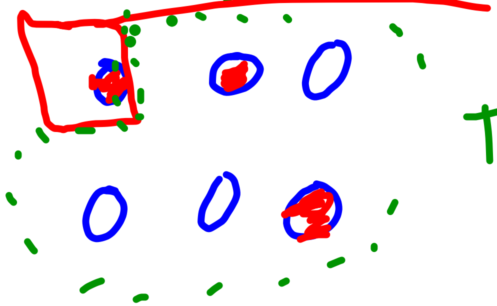
Moving to next x^*





Rejection means $x(i) = x(j) = 1$ for edge $\{i, j\}$.

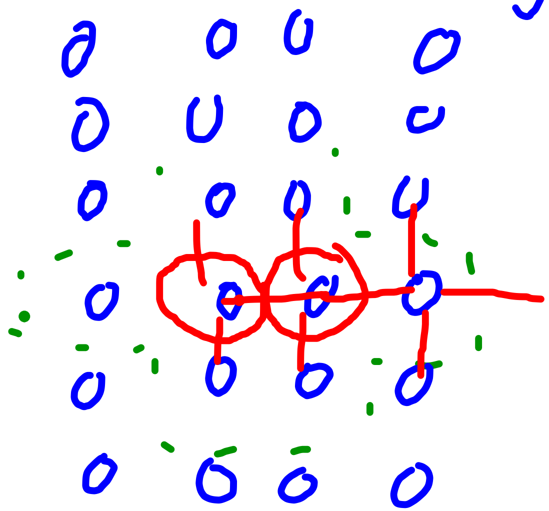
Recycling comes in here....



Stays same "recycled"

Throw away all edges, redraw those nodes

Piece of larger graph



max deg Δ

2 things could occur

- Add an edge $+1$
- Delete a bunch of edges

$$2\Delta^2 - 2$$

How many edges in X_t^*?

$$E(X_{t+1}^* \mid X_t^*) = X_t^* + 1 \cdot \left(1 - \left(\frac{\lambda}{1+\lambda}\right)^2\right) - (2\Delta^2 - 2)\left(\frac{\lambda}{1+\lambda}\right)^2 > X_t^*$$

$$1 - \left(\frac{\lambda}{1+\lambda}\right)^2 > (2\Delta^2 - 2)\left(\frac{\lambda}{1+\lambda}\right)^2$$

$$1 > (2\Delta^2 - 1)\left(\frac{\lambda}{1+\lambda}\right)^2$$

$$\frac{1+\lambda}{\lambda} > \sqrt{2\Delta^2 - 1} \qquad 1 + \lambda > \lambda\sqrt{2\Delta^2 - 1}$$

$$1+\lambda > \lambda\sqrt{2\Delta^2-1}$$

$$1 > \lambda\left(\sqrt{2\Delta^2-1}-1\right)$$

$$\boxed{\lambda < \frac{1}{\sqrt{2\Delta^2-1}-1}} = O\left(\frac{1}{\Delta}\right)$$

$$E[T] \le \frac{|E|}{\left(1-\left(\frac{\lambda}{1+\lambda}\right)^2-(2\Delta^2-1)\left(\frac{\Delta}{1+\lambda}\right)^2\right)}$$

Another way to create X_t^*

Not to build edge, build nodes

No nodes to begin w/

Add nodes in at each step...

$$\lambda < \frac{4}{3\Delta - 4}$$

Advantages over MCMC

- No mixing time
- (X^*_T, X) is absorbing

Disadvantages

- More difficult to create one of these processes than to create a Gibbs or Metropolis chain.

Hard core gas model

absorbing $X_t^* = E$

run forward until all edges in place, stop.