On a Gibbs Measure/Markov Random Field Representation for Complex Load-Sharing Systems

by

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Outline

- I. Objectives
- II. Motivation/Background
- III. Load Sharing
- IV Evolution of Component Failures
  - Phase I/II Cycles of Failures
  - Markovian Structure
- V. A Gibbs Measure/MRF Representation
  - GM/MRF Preliminaries
  - Representation for LS-systems

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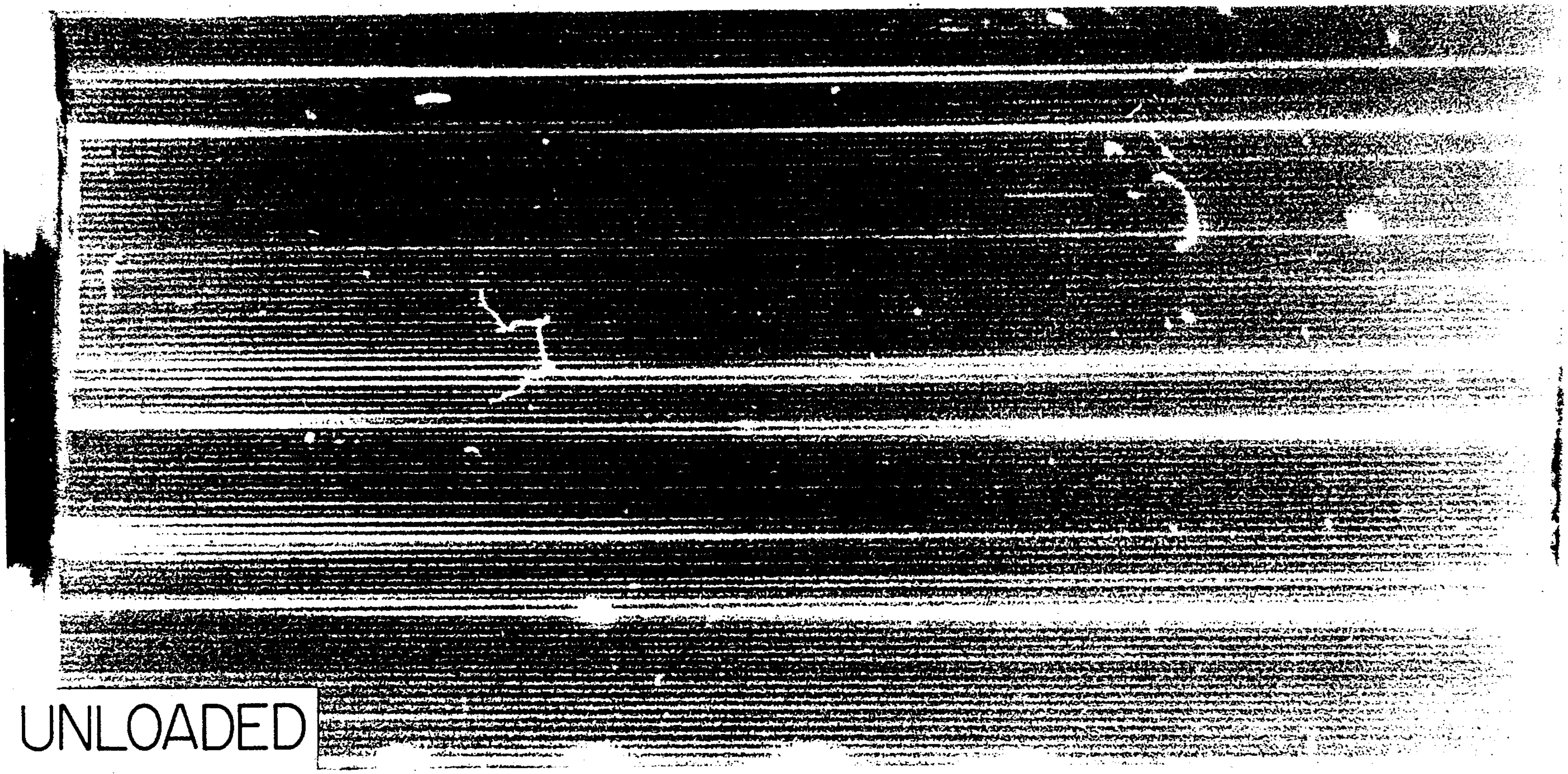
I. Some Objectives of Modeling Complex Systems

- **To model the system**
  - » **Incorporate Component (Marginal/Coupon) Information into the System Model**
  - » **Model Component Dependencies and Interactions**
  - » **Theory Should Be Realistic and Tractable**
  - » **Theory Should Be Composed of Simple Building Blocks**
- **Use "Load-Sharing" to accomplish this**

II. Motivation: Rosen's Experiment

- Single layer of parallel glass fibers
- Embedded in an epoxy material (the matrix)
- Increasing tensile load applied parallel to the fibrous composite until composite fractures
- Series of pictures identify fiber segment failures
  - Taken at various percentages of the ultimate load.

II. Motivation: Rosen's Pictures



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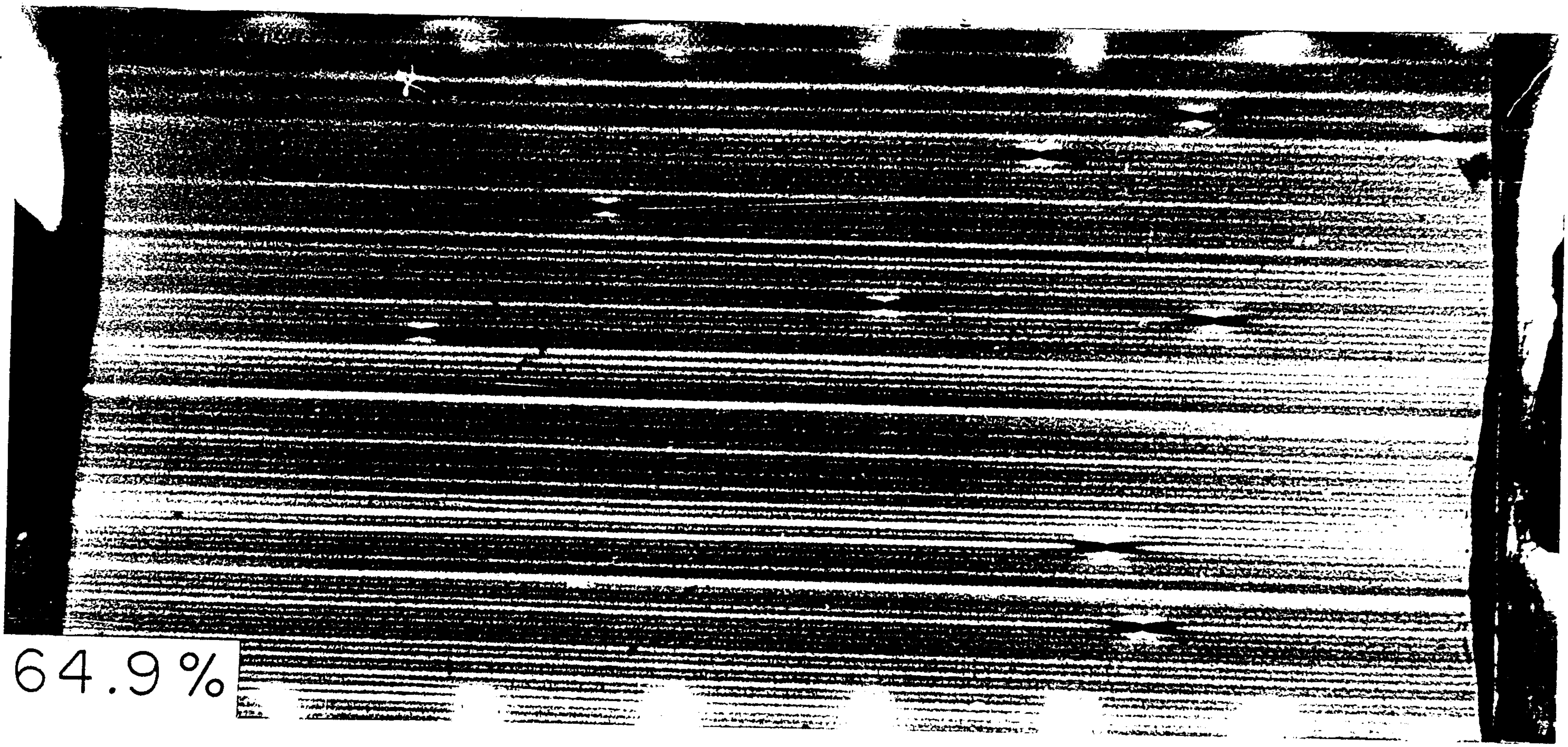

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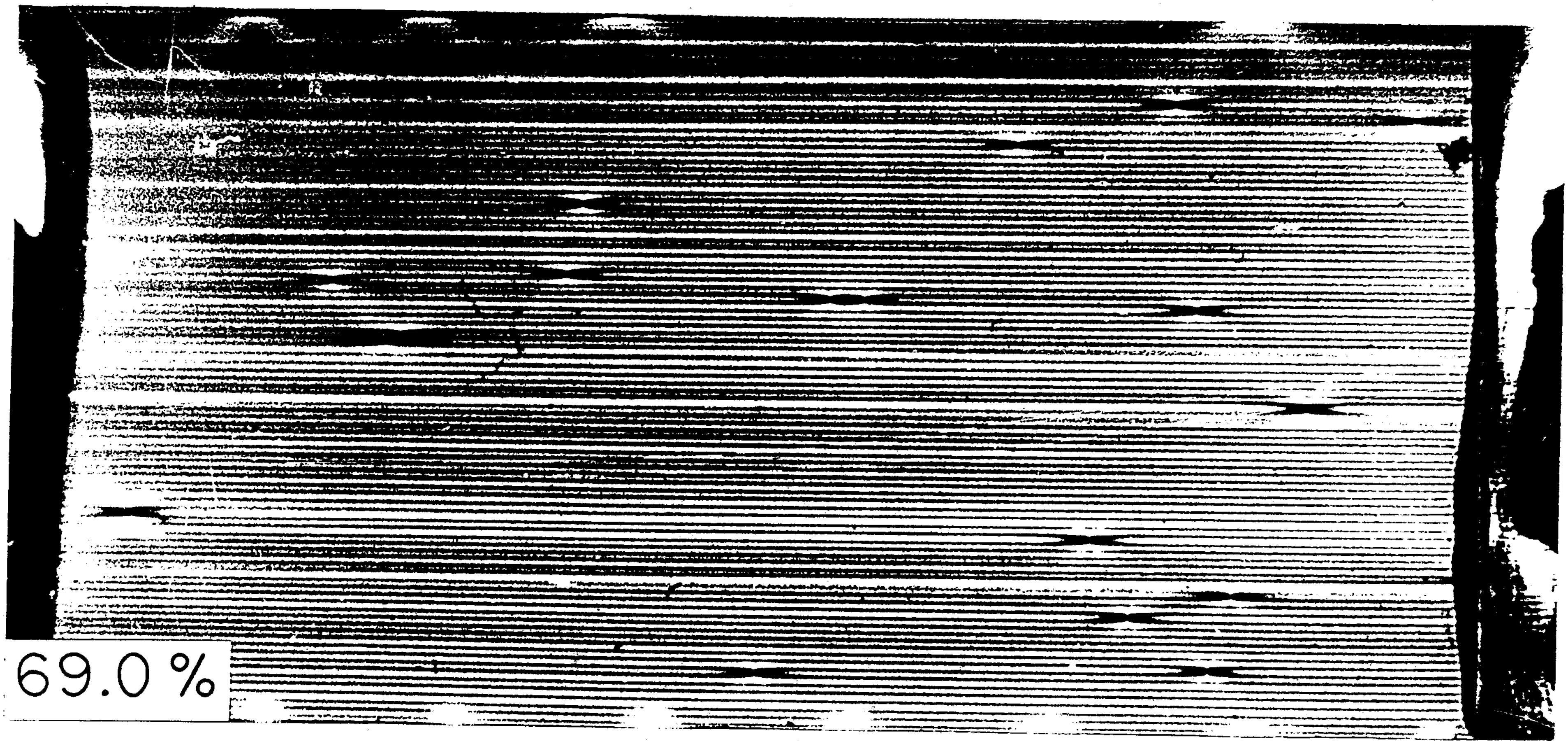
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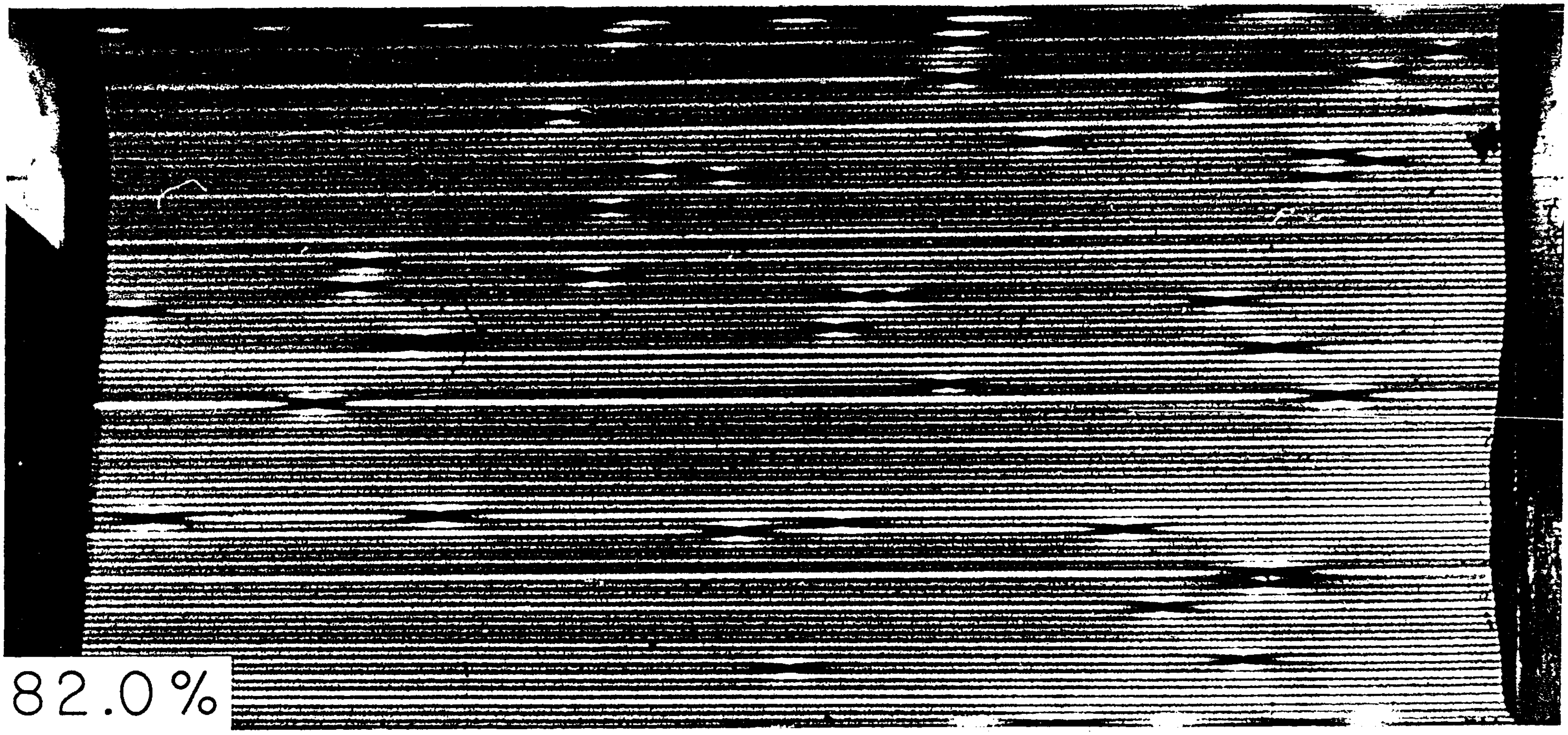
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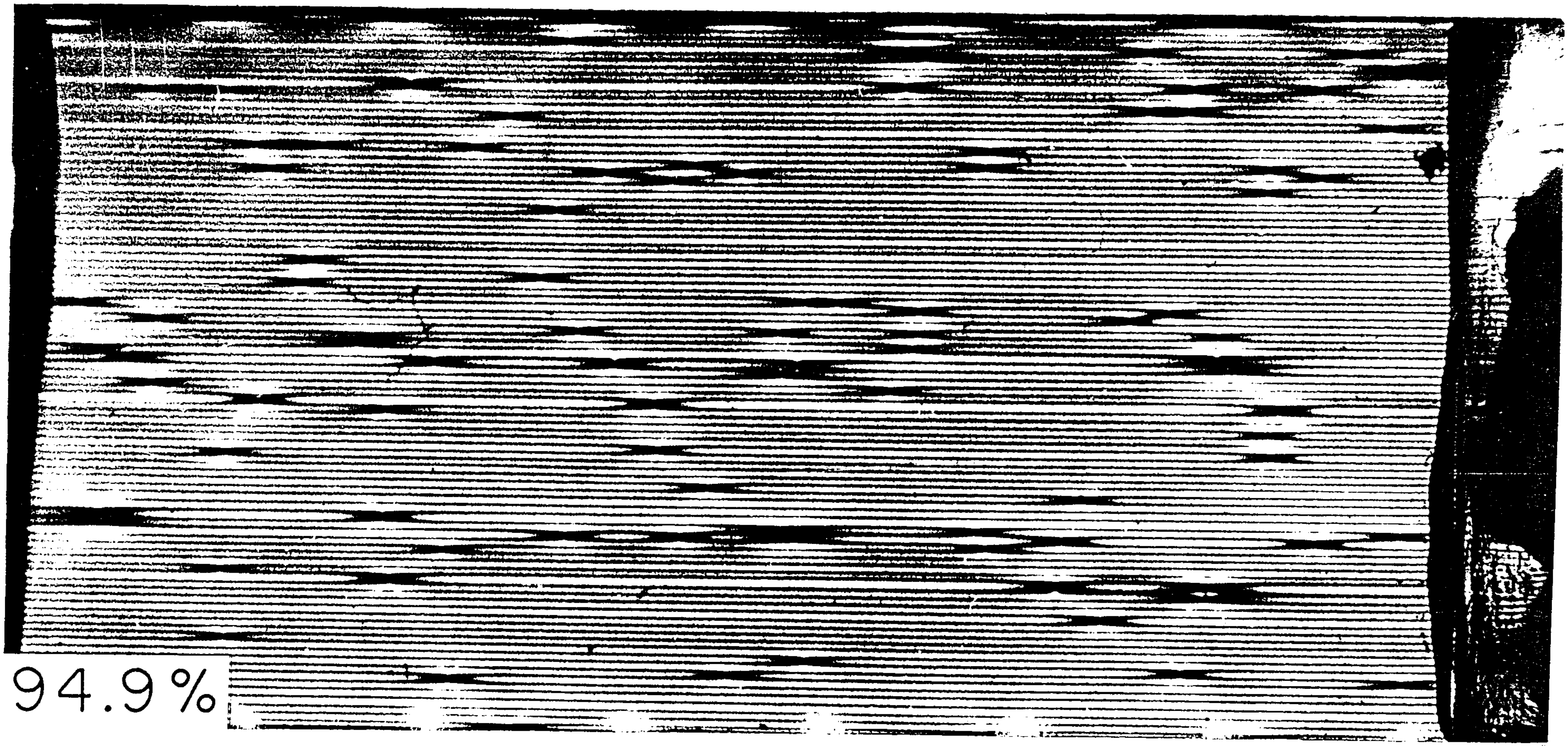
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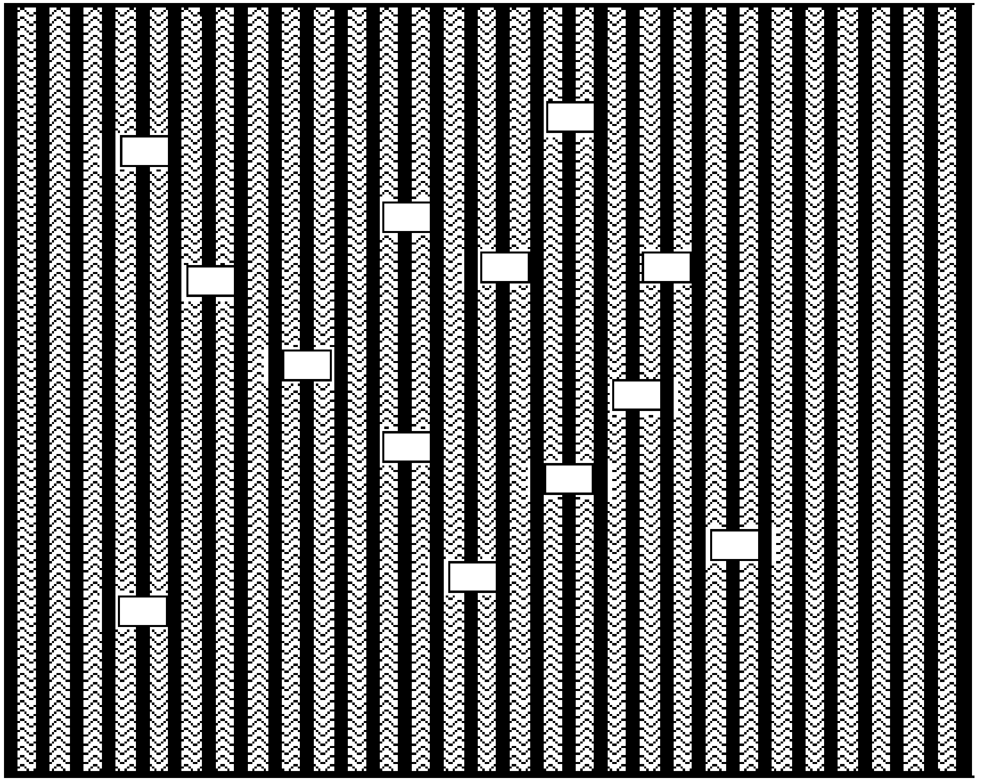
AFTER FAILURE

II. Background

Rosen's Experiment/Ineffective Length

- Rosen Discovered That There Was No Load Transfer A Certain Distance Around Breaks In Glass Fibers Embedded In A Matrix Material



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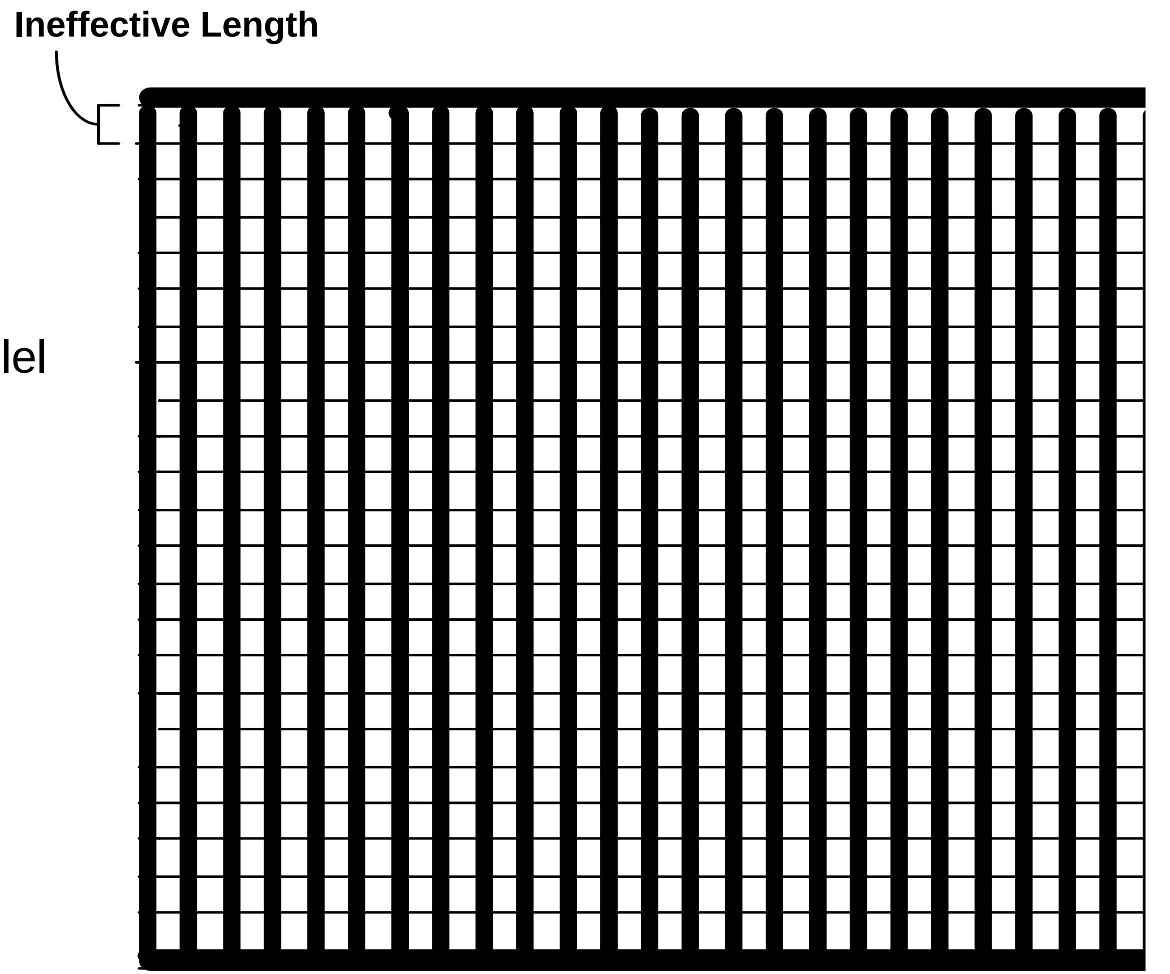
II. Background: Rosen's Model

An Idealized Model



- A System of Independent Components
    - System Is Series-Parallel
    - Series Of Bundles
    - A Bundle Is A Parallel Subsystem
- Dependencies Between Components Are Due To Load Transfer

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II. Background
Load Sharing Systems

- **Rosen's Idealized System - Chain of Bundles (Series/Parallel System)**
  - Bundle was a Parallel System Of Independent Components (Fiber Segments)
  - Bundles Independent
  - Chain Fails when a Bundle Breaks
  - Dependencies Occur Between Components Due To Load Transfer When A Component Fails (Mechanical Consideration)
  - Equal Load Sharing in Bundle (Daniels' model)
- **For Us, System is One (Big) Bundle with Monotone Load Sharing**

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III. Load Sharing Rules
Harlow, Smith and Taylor (1983)

- ***Load Sharing System***
  - ***n*** components in a system, $N=\{1,2,\ldots n\}$
  - A component load is based on the ***nominal load per component*** (the load per component or *stress*), say *s*.
    - If $M \subseteq N$ denotes the set of working components in *N*, then the load at component *i* in *M* for a nominal load per component *s* is give by $\lambda_i(M)s$
- ***Load Sharing Rule***
  - The non-negative constants, $\lambda_i(M)$, define the ***load sharing***
  - The collection $\{\lambda_i(M) : i \in M, M \subseteq N\}$ is called a ***load sharing rule***

III. Montone Load Sharing Rules

System Strength

- A load sharing rule is referred to as a ***monotone load sharing rule*** if

  – $\lambda_i(L) \geq \lambda_i(M)$ for all $i \in L, L \subseteq M \subseteq N,$

  – $\sum_{i \in M} \lambda_i(M) > 0$ for all $M \subseteq N.$

- ***System Strength*** is given by where X_i is the strength of the i^{th} component. P^* is set of path sets for system

$$S = \max_{M \in P^*} \min_{i \in M} \frac{X_i}{\lambda_i(M)}$$

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IV. Evolution of Component Failures

Phase I/II cycles of failures

- Under increasing load the system incurs component failures as follows:
  - The system incurs a series of K Phase I failures due to load sharing at nominal loads per components $S_1 < S_2 < \ldots < S_K = S$
  - The set of working components just prior to breaking stress S_i is A_i. Note that A_1=N. Let $X_{\underline{u(i)}}$ denote the component strength of the component that fails at S_i. Then $X_{U(i)} = \lambda_{U(i)}(A_i)S_i$.
  - When component U(i) fails, sudden load transfer causes a series of component failures (Phase II failures). The set of working components after these failures is A_{i+1}.

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IV. Evolution of Component Failures

Phase I/II cycles of failures – Markovian Structure

- Under monotone load sharing and independent components
  - $S_m \mid S_i\ i \le m-1,\ A_i\ i \le m =_d S_m \mid S_{m-1}, A_m$

$$(S_m = \min_{i \in A_m} \frac{X_i}{\lambda_i(A_m)} > S_{m-1})$$

  - $U(m) \mid S_i\ i \le m,\ A_i\ i \le m,\ U(i)\ i \le m-1 =_d U(m) \mid S_m, A_m$
  - $A_{m+1} \mid S_i, A_i\ i \le m,\ U(m) =_d A_{m+1} \mid S_m, A_m, U(m)$

    and is a Markov Random Field on A_m with Gibbs measure

$$P_{S_m, A_m, U(m)}(A),\ A \subseteq A_m - U(m)$$

  - $\{(A_1, S_1, U(1)), \{(A_2, S_2, U(2)), \ldots\}$ is a Markov chain.

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IV. Evolution of Component Failures

Phase I/II cycles of failures – Markovian Structure

1. $$S_m = \min_{i \in A_m} \frac{X_i}{\lambda_i(A_m)} \to U(m) \to A_{m+1} \subseteq A_m - \{U(m)\}$$

$$A_{m+1} \to S_{m+1} \to U(m+1) \to A_{m+2} \to \ldots$$

2. $U(m)$'s distribution is determined by the failure rates at S_{m} of the components in A_{m}

3. $P_{S_m, A_m, U(m)}(A),\ A \subseteq A_m - \{U(m)\}$ is the GM

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V. GM/MRF Representation *Preliminaries*

- **Finite Undirected Graph G=(N,E)**
  - **N={1,2,…,n} set of n nodes, E=set of edges**
  - **Edges determine neighbors, δi denotes the neighbors of node i**
  - **Cliques are sets of nodes all of which are neighbors of one another**
- **MRF on G**
  - **$Y_1,\ldots,Y_n$ node rv's taking values in some finite set S**
  - **For $\omega_k \varepsilon S$, $P(Y_i = \omega_i \mid Y_j = \omega_j, j \varepsilon N-\{i\}) = P(Y_i = \omega_i \mid Y_j = \omega_j, j \varepsilon \delta i)$**
- **GM on G**
  - **Potentials: $V_A(\underline{\omega})$, $\underline{\omega}$ ε S, A a subset of N**
  - **Energy:** $U(\underline{\omega}) = -\sum_{A \subseteq N} V_A(\underline{\omega})$
  - **Partition function:** $Z = \sum_{\underline{\omega} \in S^n} \exp\{-U(\underline{\omega})\}$
  - **GM:** $P(\underline{Y} = \underline{\omega}) \equiv P(\underline{\omega}) = \dfrac{\exp\{-U(\underline{\omega})\}}{Z}$

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V. GM/MRF Representation
Preliminaries

- **Canonical Energy and Potentials:**

$$\tilde{U}(\underline{\omega}) \equiv U(\underline{\omega}) - U(\underline{0}) \text{ and } \tilde{V}_A(\underline{\omega}) \equiv \sum_{B \subseteq A} -1^{|A-B|}\, \tilde{U}(\underline{\omega}^B)$$

- **Nearest Neighbor Potentials:**

$$\tilde{V}_A(\underline{\omega}) \text{ is a NN potential and } \tilde{U}(\underline{\omega}) = -\sum_{A \subseteq N} \tilde{V}_A(\underline{\omega})$$

1. NN Potential means $\tilde{V}_A$ vanishes if A is not a clique
2. $\tilde{V}_A(\underline{\omega}) = 0$ if $\omega_i = 0$ for some $i \in A$.

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V. GM/MRF Representation *for LS system Static Case*

- **MRF on directed graph G*=(N,E*)**
  - **$Y_1,\ldots,Y_n$ node rv's taking values in S={0,1}**
  - **Let I_i be the set of indices in the smallest collection of rv's such that**

    $$P(Y_i = 1 \mid Y_j = \omega_j, j \in N-\{i\}) = P(Y_i = 1 \mid Y_j = \omega_j, j \in I_j)$$

  - **Definitions:**

    $*$ j influences i $(j \rightarrow i)$ if $j \in I_i$.

    $*$ Sites i and j are called neighbors if either $j \rightarrow i$ or $i \rightarrow$ j

    $*$ The set $N_i = \{j : j \rightarrow i \text{ or } i \rightarrow j\}$.

  - **Consequences: Since N_j contains I_j,**

    $$P(Y_i = 1 \mid Y_j = \omega_j, j \in N-\{i\}) = P(Y_i = 1 \mid Y_j = \omega_j, j \in N_j) = P(Y_i = 1 \mid Y_j = \omega_j, j \in I_j)$$

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V. GM/MRF Representation *for LS system*
Static Case: Local Structure of MRF

- **Log odds ratio for node i:**

$$\sigma_i(A,s) \equiv \frac{P(Y_i = 1 \mid Y_j = 1, j \in A - \{i\}, Y_k = 0, k \in A^c)}{P(Y_i = 0 \mid Y_j = 1, j \in A - \{i\}, Y_k = 0, k \in A^c)} \quad (A)$$

$$= \frac{P(Y_i = 1, Y_j = 1, j \in A - \{i\}, Y_k = 0, k \in A^c)}{P(Y_i = 0, Y_j = 1, j \in A - \{i\}, Y_k = 0, k \in A^c)} \quad (B)$$

- **Let F_i denote the strength distribution of component i and S_i = 1- F_j the strength survival distribution. From (A),**

$$\sigma_i(A,s) = \ln \frac{S_i(\lambda_i(A)s)}{F_i(\lambda_i(A)s)} \quad (C)$$

while from (B) and the canonical potential representation for GM's

$$\sigma_i(A,s) = \sum_{\{K \subseteq A, i \in K\}} V_K(\underline{\omega}(A), s) \quad (D)$$

where $\underline{\omega}(A)$ is the vector of component states with

$\omega_i = 1$ if $i \in A$ and $= 0$ if $i \notin A$.

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V. GM/MRF Representation *for LS system*
Log Odds is the Primitive

- **Consequences of (A)-(D)**
  - **(A) and (C) imply that the LS-Rule determines G***
  - **(C) and (D) with the Mobius inversion formula give**

$$V_K(\underline{\omega}(K),s) = \frac{\sum_{A \subseteq K} \sum_{i \in A} (-1)^{|K-A|} \sigma_i(A,s)}{|K|}$$

  - **In the dynamic case, replace the component strength distributions by conditinal ones, conditional on $X_i > \lambda_i(A_m)S_m$**

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Concluding Comments

- **Simple formulas if the component strength distributions are log-logistic. Useful in understanding the system structure.**
- **Static case does not depend on monotone LS to get the GM/MRF**
- **Conjecture that above rep works for arbitrary LS. Just get recursive formulas for the conditional distributions that depend on the whole history.**
- **Can generate an "exact" sample of system failure which can involve lots of comparisons of component strengths depending on the asymmetry and spatial dependence of the LS rule.**
- **Doubly Intractable Issues (Iain Murray talk at Duke in January 2009)**

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