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ON A GIBBS MEASURE REPRESENTATION FOR COMPLEX LOAD-SHARING PARALLEL SYSTEMS

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Abstract

Complex load-sharing parallel systems incorporate dependencies between components through a load-sharing rule. Such systems were used to model the strength of fibrous composites where component fiber failures occur in cycles having two phases. As the load on the system increases, a single component fails (Phase I), which then causes a cascade of component failures (Phase II) due to the load transfer as these Phase II failures occur. Under monotone load-sharing the failures have a simple Markovian structure. The current state of the system only depends on the latest Phase I failure and the corresponding Phase II failures have a Gibbs measure (GM) representation when the component strengths are independent. The GM representation formalism indicates that log odds ratios for the strength distribution are the fundamental quantities in determining the potentials in the GM for the state of the system. Formulas for the potentials are particularly simple through the use of generalizations of the log-logistic and the log-logistic distributions for the component strength distributions. These simplified formulas can be used to understand the system structural reliability.

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1. Introduction

Modern equipment or materials are complex systems where statistically modeling the system is critical for assessing system reliability. In classical system reliability, the components are independent. This simplification allows one to develop a rich reliability theory for such systems. (See for example Barlow and Proschan, 1975, and Samaniego, 2007).

One aspect of this theory is Samaniego's representation (1985) of a system of iid components in terms of a mixture over the order statistics distributions where the vector of mixing proportions is referred to as the ***system signature***. System signatures have been used by Samaniego and his co-authors (Kochar et al, 1999 and Samaniego, 2007) to compare systems and networks and address what we consider properties of the ***system structure***.

In many systems the components are not independent but consist or can be viewed as a structure composed of dependent or interacting components. Since it would not be unusual to have testing information regarding individual component behavior outside the system, an important question is how this is incorporated into the model to assess system reliability where the components are dependent.

Here we study load-sharing parallel systems that incorporate dependencies into the system in such a way that they can address this question. Such a system was first considered as a model for the strength of a bundle of threads (H. E. Daniel, 1945), but was studied extensively by Rosen (1964, 1965) in the context of fibrous composites. These systems are discussed in Section 3 where we obtain a Gibbs Measure/Markov Random Field representation in Section 3.1 for the system state for static systems. The fundamental building blocks for such a system are the log odds ratios of the individual component strengths. In particular, the log odds define the local structure of the Markov random field. Formulas are also given to show how the potentials that define the Gibbs measure for the field are simple functions of these log odds. These formulas are particularly simple when the component strength distributions are log-logistic and give insight into the system structure even when the strength distributions are not log-logistic.

The Markovian structure of dynamic monotone load-sharing systems is described in detail in Section 3.2, where the representation for the system state is obtained using the result for the static case. The relationship of system state to the system structure is discussed in Section 4. In Section 5, we discuss some simple examples, including Daniels' equal load-sharing model and the local load-sharing model, to illustrate the results from the preceding sections and show how the state probability can be interpreted in terms of the potentials. The paper concludes with some closing comments regarding Gibbs measure representations for load-sharing systems. Given an undirected graph G and a load-sharing rule, it is shown how to construct a load-sharing rule where the load sharing system is a Markov random field whose neighborhood structure is given by G. In the next section, we present some background material regarding classical systems and Gibbs Measure/Markov Random Field representations. In particular, the Gibbs measure representation for a classical system leads to terms that can be used to describe the complexity of the system structure.

2. Classical reliability systems

The starting point in system reliability theory is the system structure where the system is described in terms of cut and path sets. From the system structure the static system reliability, $h(\underline{p})$, is determined when the components are independent Bernoulli random variables that indicate whether or not the component works. Time is incorporated into the system reliability by replacing p_i by $\overline{F}_i(t)$, the survival probability that component's lifetime exceeds t. In Section 2.1, we consider the case of static systems of iid components. Section 2.2 discusses Gibbs measure representations related to system reliability.

2.1. Static systems with i.i.d. components

Consider the reliability of a system of n iid Bernoulli components where the probability that the i^{th} component works is p and the probability it fails is $\overline{p} = 1 - p$. Then the reliability of the system is denoted by $h(p)$ and the probability that the system fails is $\overline{h(p)} = 1 - h(p)$.

For deriving some representations for $h(p)$, we will consider the following scenario that is motivated by Boland's (2001) work on signatures for majority systems. Before doing this, note that system signatures have been studied extensively to analyze system structure (see Samaniego, 2007, and the references therein). Samaniego (1985) was the first to introduce the concept of signatures where the vector with components $s_i = S_i/n!, i = 1, \ldots, n$ defined below is referred to as the signature of the system.

Definition A subset $C \subseteq \{1, \ldots, n\}$ of components is called a ***cut set*** if the failure of all the components in C causes the system to fail. A subset P of components is called a ***path set*** if the functioning of all the components in P implies that the system functions. An ordered collection of components $(\pi_1, \ldots, \pi_k)$ is called an ***ordered cut set*** if the set $(\pi_1, \ldots, \pi_k)$ is a cut set. It is a ***minimal ordered cut set*** if $(\pi_1, \ldots, \pi_{k-1})$ is not an ordered cut set.

The scenario we consider here is as follows. Consider a system where all the components work. Generate a binomial random variable N with parameters (n, p), and independent of N generate a random permutation $\underline{\Pi} = (\Pi_1, \ldots, \Pi_n)$ where all permutations are equally likely. If $N = i$, serially replace components Π_1 through Π_{n-i} with failed components. After the replacement, note that the component reliabilities under this scenario are iid Bernoulli with the probability a given component works equal to p.

Following Boland (2001) let A_i be the number of path sets of size i and let $a_i = \frac{A_i}{\binom{n}{i}}$. The proportion a_i is just the proportion of component subsets of size i that are path sets. Note that $a_n = 1$. Let D denote whether or not the system has failed ($D = 0$ if the system failed and $D = 1$ otherwise). Then,

$$P(N = i, D = 1) = P(D = 1|N = i)P(N = i) = a_i \binom{n}{i} p^i \overline{p}^{n-i} \qquad (2.1a)$$

and

$$h(p) = P(D = 1) = \sum_{i=1}^{n} a_i \binom{n}{i} p^i \overline{p}^{n-i} \tag{2.1b}$$

The identity in (2.1b) is the representation of $h(p)$ in terms of the path set proportions a_i. To derive the representation in terms of signatures, let S_i denote the number of orderings for which the i^{th} serial failure causes system failure, i.e, orderings $\underline{\pi} = (\pi_1, \ldots, \pi_n)$ of n components where $(\pi_1, \ldots, \pi_i)$ is the minimal ordered cut set for $\underline{\pi}$. Let $M(\underline{\pi}) = i$ when $(\pi_1, \ldots, \pi_i)$ is the minimal ordered cut set for $\underline{\pi}$ and let $M \equiv M(\underline{\Pi})$. Note that M and N are independent since $\underline{\Pi}$ and N are independent. Let $s_i = S_i/n!, i = 1, \ldots, n$. Then, $P(M = i) = s_i$. As noted earlier, the vector $(s_1, \ldots, s_n)$ is referred to as the ***system signature***.

With the above formulation and for a given configuration $\underline{\pi}$, the system works if and only if $N > n - i$ when $(\pi_1, \ldots, \pi_i)$ is the minimal ordered cut set for $\underline{\pi}$. Thus,

$$P(M = i, D = 1) = P(M = i, N > n - i) = s_i \sum_{j=n-i+1}^{n} \binom{n}{j} p^j \overline{p}^{n-j} \tag{2.2a}$$

and

$$h(p) = \sum_{i=1}^{n} s_i \sum_{j=n-i+1}^{n} \binom{n}{j} p^j \overline{p}^{n-j} = \sum_{j=1}^{n} (\sum_{i=n-j+1}^{n} s_i) \binom{n}{j} p^j \overline{p}^{n-j} \tag{2.2b}$$

As noted by Boland (2001), equating the last terms in (2.1b) and (2.2b) gives

$$a_j = \sum_{i=n-j+1}^{n} s_i \tag{2.3}$$

Identity (2.3) shows that a_j is increasing in j.

Finally, let $M_i = S_i/[(n-i)!(i-1)!]$ which is the number of ways that the i^{th} component failure causes system failure and the component that causes system failure is specified as well as the set of $i - 1$ other components that have failed. Thus,

$$M_i = s_i \binom{n}{i} i. \tag{2.4}$$

and

$$P(M = i, N = n - i, D = 1) = M_i p^i \overline{p}^{n-i} = s_i \binom{n}{i} i p^i \overline{p}^{n-i}$$

Identities (2.3) and (2.4) summarize the path and cut set relationships with the system signature.

2.2. Gibbs Measure Representations

Here we consider a ***Gibbs Measure (GM)*** representation for the states of the components. Since the components in a classical system are independent, the associated graph for the ***Markov Random Field (MRF)*** corresponding to the Gibbs measure for the states has no edges because the edges denote the dependencies among components. Even in this elementary case, the Gibbs measure perspective gives some interesting insights into the system structure. In particular, the concept of the complexity of the structure is part of the Gibbs measure representation for the joint distribution of the component states and the state of the system and has information theoretic implications.

Below we give the necessary background regarding to the GM/MRF representation. The formulation is based on Sandberg (2004), (see also Grimmett, 1973).

Let $G = (N, E)$ be a finite graph where $N = \{1, \ldots, n\}$ is the set of sites and E denotes the set of edges in the graph. We call sites $i, j \in N$ ***neighbors*** in the graph if there is an edge between them. For a set $A \subset N$, let ∂A denote its neighbor (or boundary) set: all elements in $N \backslash A$ that have a neighbor in A. For $i \in N$, let $\partial i = \partial\{i\}$. A ***clique*** is a set of sites where all the sites are neighbors of one another.

Let $Y_1, \ldots, Y_n$ be random variables taking values in some finite set S. The random variables are said to define a ***Markov Random Field (MRF)*** on the graph G if, for any vector $\underline{\omega} \in S^n$,

$$P(Y_i = \omega_i \mid Y_j = \omega_j, j \in N - \{i\}) = P(Y_i = \omega_i \mid Y_j = \omega_j, j \in \partial i) \tag{2.5}$$

The ***Gibbs Measure (GM)*** representation for the joint distribution of $\underline{Y} = (Y_1, \ldots, Y_n)$

will be defined based on the following concepts.

Fundamental to the GM representation is the notion of ***potential***, which is just a function on S^n indexed by sets $A \subseteq N$. For a set of sites $A \subseteq N$, the potential of $\underline{\omega}$ is denoted by $V_A(\underline{\omega})$.

Given a potential, the ***energy*** of $\underline{\omega}$ will be defined as

$$U(\underline{\omega}) = -\sum_{A \subseteq N} V_A(\underline{\omega}). \tag{2.6a}$$

Using the energy, we define the Gibbs measure P as

$$P(\underline{Y} = \underline{\omega}) = P(\underline{\omega}) = \frac{exp(-U(\underline{\omega}))}{Z}, \tag{2.6b}$$

where Z is the ***normalizing constant***

$$Z = \sum_{\underline{\omega} \in S^n} exp(-U(\underline{\omega})). \tag{2.6c}$$

The normalizing constant Z is referred to as the ***partition function*** in statistical mechanics.

It is implicit in (2.6a-c) that the Gibbs measure satisfies the ***positivity condition***:

$$P(\underline{\omega}) > 0 \text{ for all } \underline{\omega} \in S^n. \tag{2.7}$$

The Gibbs measure can be associated to the MRF defined by $\underline{Y}$ on the graph G through a special type of potential which is called the nearest neighbor potential. We call $V_A(\underline{\omega})$ a ***nearest neighbor potential*** if $V_A(\underline{\omega})$ is identically zero whenever A is not a clique as defined by the MRF on G. Note that the nearest neighbor potential defines the graph $G(N, E)$ because it gives the set of cliques in G. This totally determines the edges between sites since the edges are just the set of cliques of size 2. As we shall see below, there is an equivalence between the Gibbs measure that is defined by the potential V_A through (2.6a-c) and the laws of the MRF if and only if V_A is a nearest neighbor potential.

There is not a unique energy or set of potentials associated with a Gibbs measure, but we can define a special energy and class of potentials that is unique as follows.

Given some preferred element in S^n, denoted here by $\underline{0}$, we define the ***canonical energy*** of a probability measure P as

$$\tilde{U}(\underline{\omega}) = -(logP(\underline{\omega}) - logP(\underline{0})) = U(\underline{\omega}) - U(\underline{0}) \tag{2.8a}$$

The ***canonical potential*** of measure is then defined by

$$\tilde{V}_A(\underline{\omega}) = \sum_{B \subseteq A} -1^{|A-B|} \tilde{U}(\underline{\omega}^B) \tag{2.8b}$$

where $\underline{\omega}^B$ is the configuration which takes the same value as $\underline{\omega}$ on B, but sets all values to 0 elsewhere. Formulas (2.6a) and (2.8b) are just the Mobius inversion formulas that relate $\tilde{U}$ and $\tilde{V}$ (See Berge, 1971, Section 3.2).

As indicated by the next three lemmas and theorem, the potential $V_A(\underline{\omega})$ defined by (2.8b) corresponds to the probability measure P, and is indeed a nearest neighbor potential for the MRF defined in (2.5).

Lemma 2.2.1 Any probability measure satisfying (2.7) is a Gibbs measure with the form

$$P(\underline{\omega}) = \frac{exp(-\tilde{U}(\underline{\omega})}{Z} \tag{2.9a}$$

where $\tilde{U}$ is the canonical energy defined in (2.8a) and is related to the canonical potential via (2.6a) and (2.8b).

Remark: Note that in the canonical energy representation, $Z^{-1} = P(\underline{0})$.

Lemma 2.2.2 Given a Gibbs measure with canonical representation (2.9a), there is a corresponding MRF where its neighborhood structure is generated by the nearest potential.

Lemma 2.2.3 If P is the joint distribution of a MRF, then the corresponding canonical potential $\tilde{V}_A(\underline{\omega})$ is a nearest neighbor potential and

$$\tilde{U}(\underline{\omega}) = -\sum_{A \subseteq N} \tilde{V}_A(\underline{\omega}) \tag{2.9b}$$

Theorem 2.2.1 Let P be a joint distribution defined on a space S^n satisfying (2.7). The following are equivalent:

• P is the joint distribution of a MRF.

• P is the joint distribution on S^n with unique canonical potentials $\tilde{V}_A(\underline{\omega})$, which is a nearest neighbor potential.

A simple consequence of Lemma 2.2.3 is the following property of nearest neighbor potentials.

Lemma 2.2.4 If $\tilde{V}_A(\underline{\omega})$ is a nearest neighbor potential, then $\tilde{V}_A(\underline{\omega}) = 0$ whenever $\omega_i = 0$ for some $i \in A$.

In reliability we are oftentimes interested in systems of components where $S = \{0, 1\}$ where 0 indicates that the component has failed and 1 indicates that it works. For 0-1 systems, the potentials are simple functions of the local structure (2.5) of the field. In this case, the local structure for the Markov random field on the subsets of N is the log odds ratio of component i working to not working given that the components in $A\{i\}$ work and those in A^c do not when the system is under load s. So,

$$\begin{aligned} \sigma_i(A, s) &\equiv ln \frac{P(Y_i = 1 | Y_j = 1, j \in A - \{i\}, Y_k = 0, k \in A^c)}{P(Y_i = 0 | Y_j = 1, j \in A - \{i\}, Y_k = 0, k \in A^c)} \\ &= ln \frac{P(Y_i = 1, Y_j = 1, j \in A - \{i\}, Y_k = 0, k \in A^c)}{P(Y_i = 0, Y_j = 1, j \in A - \{i\}, Y_k = 0, k \in A^c)} \end{aligned} \tag{2.10}$$

For $A \subseteq N$, define $\underline{\omega}(A) \in \{0,1\}^n$ by

$$\underline{\omega}(A) = ((\omega_1, \ldots, \omega_n) : \omega_i = 1, i \in A, \omega_j = 0, j \in A^c).$$

As before in this section, ω_i just indicates whether or not component i is working. Then we can write $P_s(A)$, the probability that under load s the working set is A, as a Gibbs measure in the form

$$P_s(A) = P_s(\underline{\omega}(A)) = \frac{1}{Z(s)} exp\{-U(\underline{\omega}(A), s)\} \tag{2.11}$$

where we choose $U(\underline{\omega}(\emptyset), s) = 0$, when the set of working components is $\emptyset$. In this case, $P_s(\emptyset) = \frac{1}{Z(s)}$ and U is the canonical energy.

By the previous discussion, $U(\underline{\omega}(A), s) = -\sum_{K \subseteq A} V_K(\underline{\omega}(A), s)$, where $V_K(\underline{\omega}(A), s)$ is the canonical potential for the clique K at load s. Note that we sum over only cliques in set A since $V_k(\underline{\omega}(A), s) = 0$ if K is not a clique or if $K \not\subseteq A$ (see the definition of the ***nearest neighbor*** potential and Lemma 2.2.4 in this section). Thus, by writing the energy in terms of potentials, we have

$$P_s(A) = P_s(\underline{\omega}(A)) = \frac{1}{Z(s)} exp\{\sum_{K \subseteq A} V_K(\underline{\omega}(A), s)\} \tag{2.12}$$

Thus, by (2.10), we have

$$\begin{aligned} \sigma_i(A, s) &= ln \frac{P(\omega_i = 1, \omega_j = 1, j \in A - \{i\}, \omega_k = 0, k \in A^c)}{P(\omega_i = 0, \omega_j = 1, j \in A - \{i\}, \omega_k = 0, k \in A^c)} \\ &= ln \frac{P_s(A)}{P_s(A - \{i\})} \\ &= \sum_{\{K \subseteq A : i \in K\}} V_K(\underline{\omega}(A), s) \end{aligned} \tag{2.13}$$

Note that to derive (2.13), we used the fact that

$$V_K(\underline{\omega}(A), s) = V_K(\underline{\omega}(A - \{i\}), s), \text{ for } K \subseteq A - \{i\}.$$

Identity (2.13) gives a way to calculate the log odds ratios by summing up the potentials, however, we are more interested in the reverse, i.e., obtaining the potentials through the log odds ratios. The following theorem allows us to do so.

Theorem 2.2.2 The potentials $V_K(\underline{\omega}(K), s)$ can be obtained via the Mobius inversion formula as

$$V_K(\underline{\omega}(K), s) = \frac{\sum_{A \subseteq K} \sum_{i \in A} (-1)^{|K-A|} \sigma_i(A, s)}{|K|}. \tag{2.14}$$

Proof: Let $\sigma(A, s) \equiv \sum_{i \in A} \sigma_i(A, s)$. Then

$$\begin{aligned} \sigma(A, s) &= \sum_{i \in A} \sigma_i(A, s) \\ &= \sum_{i \in A} \sum_{\{K \subseteq A : i \in K\}} V_K(\underline{\omega}(A), s) \\ &= \sum_{K \subseteq A} |K| V_K(\underline{\omega}(A), s). \end{aligned} \tag{2.15}$$

Note that for any clique $K \subseteq A$ and set $B \subseteq K$, $(\underline{\omega}(A))^B = (\underline{\omega}(K))^B$, thus, by the Mobius inversion formula (Berge, 1971, see also Grimmett, 1973),

$$\begin{aligned} V_K(\underline{\omega}(A), s) &= \sum_{B \subseteq K} (-1)^{|K-B|} U((\underline{\omega}(A))^B, s) \\ &= \sum_{B \subseteq K} (-1)^{|K-B|} U((\underline{\omega}(K))^B, s) \\ &= V_K(\underline{\omega}(K), s). \end{aligned} \tag{2.16}$$

So $\sum_{K \subseteq A} |K| V_K(\underline{\omega}(A), s) = \sum_{K \subseteq A} |K| V_K(\underline{\omega}(K), s)$, since $V_K(\underline{\omega}, s)$ is a function of $\underline{\omega}$ only through the values of $\omega_i, i \in K$. Therefore,

$$\sigma(A, s) = \sum_{K \subseteq A} |K| V_K(\underline{\omega}(K), s) \tag{2.17}$$

Applying the Mobius inversion formula to (2.17), we get

$$|K| V_K(\underline{\omega}(K), s) = \sum_{A \subseteq K} (-1)^{|K-A|} \sigma(A, s) = \sum_{A \subseteq K} \sum_{i \in A} (-1)^{|K-A|} \sigma_i(A, s)$$

from which (2.14) follows immediately. □

Example **iid n-component systems**

For an iid n-component system, each component is a clique itself and there are no other cliques. Suppose the probability for an individual component working is p. Let $\underline{\omega} \in \{0,1\}^n$ be the configuration of n components, and let $D_{\underline{\omega}}$ be the set of working components under configuration $\underline{\omega}$, i.e. $D_{\underline{\omega}} = \{i : \omega_i = 1\}$. Then

$$P(\underline{\omega}) = p^{|D_{\underline{\omega}}|}(1-p)^{n-|D_{\underline{\omega}}|} = \frac{exp\{-|D_{\underline{\omega}}|log\frac{1-p}{p}\}}{exp\{-n\ log(1-p)\}}$$

where $|D_{\underline{\omega}}|$ is the cardinality of set $D_{\underline{\omega}}$.

Let $U(\underline{\omega}) = |D_{\underline{\omega}}|log(\frac{1-p}{p})$ be the "energy of $\underline{\omega}$" and let $Z = exp\{-n\ log(1-p)\}$ be the normalizing constant. Then we can write $P(\underline{\omega})$ in the form $P(\underline{\omega}) = \frac{exp(-U(\underline{\omega}))}{Z}$, which is a Gibbs measure representation for the working configuration of components. It turns out that the canonical energy $\tilde{U}(\underline{\omega})$ of this Gibbs measure P is the same as $U(\underline{\omega})$, since

$$\tilde{U}(\underline{\omega}) = -(logP(\underline{\omega}) - logP(\underline{0})) = |D_{\underline{\omega}}|log(\frac{1-p}{p}) \tag{2.18}$$

To derive the canonical potential, we use (2.8b) and Lemma 2.2.4.

Lemma 2.2.4 simplifies the calculation for the canonical potentials for the iid components system since we only need to consider the set A when $A \subseteq D_{\underline{\omega}}$. And for $A \subseteq D_{\underline{\omega}}$, by (2.8b),

$$\begin{aligned}\tilde{V}_A(\underline{\omega}) &= \sum_{B\subseteq A} -1^{|A-B|}\tilde{U}(\underline{\omega}^B) \\ &= \sum_{B\subseteq A} -1^{|A-B|}|B|log(\frac{1-p}{p}) \qquad (2.19)\end{aligned}$$

It can be shown that (2.19) is zero if $|A| \geq 2$ and it is $log(\frac{1-p}{p})$ when $|A| = 1$.

Thus, we can write the Gibbs measure representation in terms of the canonical energy as

$$P(\underline{\omega}) = \frac{exp(-\tilde{U}(\underline{\omega})}{\tilde{Z}}$$

where $\tilde{Z} = \frac{1}{P(\underline{0})} = \frac{1}{(1-p)^n}$, since $\tilde{U}(\underline{0}) = 0$. $\square$

Up to this point, we have just considered the Gibbs measure on S^n which describes the state space of the components of the system. In the remainder of this section, we derive the Gibbs measure for the joint distribution of the reliability of the system (D) and a macro variable that denotes the states of the components. We consider two macro variables where the first is the "energy level" of the component states and the second is related to the system signature. In both cases, terms occur in the Gibbs measure that do not depend on the component reliability p and can be interpreted as quantifying the complexity of the system structure. We do this in two stages. The first considers the multiplicity of the states that have the same energy level or the ordered serial failure that causes system failure. The second takes into the consideration of the reliability structure of the system.

Various states of the component of the system give rise to the same canonical energy $\tilde{U} = u$. This referred to as the ***multiplicity*** or ***degeneracy*** of that energy level and is just the number of states that have level u. By (2.18), when $\tilde{U}(\underline{\omega}) = u$,

$$|D_{\underline{\omega}}| = \frac{u}{log(\frac{1-p}{p})} \equiv K(u)$$

Then,

$$\begin{aligned} P(\underline{\omega} : \tilde{U}(\underline{\omega}) = u) &= P(\underline{\omega} : |D_{\underline{\omega}}| = K(u)) = \binom{n}{K(u)} p^{K(u)}(1-p)^{n-K(u)} \\ &= \binom{n}{K(u)} \frac{(\frac{1-p}{p})^{-K(u)}}{(1-p)^{-n}} = \binom{n}{K(u)} \frac{exp\{-K(u)log(\frac{1-p}{p})\}}{\tilde{Z}} \end{aligned}$$

Thus, $\binom{n}{K(u)}$ is the multiplicity of energy level $\tilde{U}(\underline{\omega}) = u$.

Having described the state of components in a reliability system, we now need to use this Gibbs measure representation to describe the state of system. To do this, let D be the indicator of the system's working condition, i.e. $D = 1$ if the system is working and $D = 0$ otherwise. Also, let A_i be the number of path sets of size i. Then we have

$$P(\underline{\omega} : |D_{\underline{\omega}}| = i, D = 1) = A_i p^i (1-p)^{n-i} = \frac{A_i \; exp\{-i \; log(\frac{1-p}{p})\}}{(1-p)^{-n}} \tag{2.20}$$

and

$$\begin{aligned} P(\underline{\omega} : |D_{\underline{\omega}}| = i, D = 0) &= [\binom{n}{i} - A_i] p^i (1-p)^{n-i} \\ &= \frac{[\binom{n}{i} - A_i] exp\{-i \; log(\frac{1-p}{p})\}}{(1-p)^{-n}} \end{aligned} \tag{2.21}$$

Let $B_i = \binom{n}{i} - A_i$ denote the ***number of cut set*** of size $n - i$, and let $K \equiv K(\tilde{U}(\underline{\omega})) \equiv |D_{\underline{\omega}}|$. Then for any $\underline{\omega}$ where $K(\tilde{U}(\underline{\omega})) = k$ and $D = d$, we have from (2.20) and (2.21) that

$$\begin{aligned} U(k,d) &= \tilde{U}(\underline{\omega}) - (d \; logA_i + (1-d) logB_i) \\ &= k \; log(\frac{1-p}{p}) - (d \; logA_k + (1-d) logB_k) \end{aligned} \tag{2.22}$$

Then the Gibbs measure representation for the joint distribution of (K, D) is

$$P(K = k, D = d) = P(k, d) = \frac{exp\{-U(k,d)\}}{Z},$$

where $Z = (1-p)^{-n}$.

When we consider the system's working condition $D = d$ in addition to the states of components, the canonical energy was modified by the system structure. This difference between $\tilde{U}(\underline{\omega})$ and $U(k,d)$ is

$$T(k, d; \underline{A}, \underline{B}) = d \; logA_k + (1-d) logB_k,$$

We refer to $T(k, d; \underline{A}, \underline{B})$ as the ***complexity of the structure***.

Note that the state space for the vector (K, D) is a subset of $S = \{0, 1, \ldots, n\} \times \{0, 1\}$. Since some of the points in this product space must have probability zero (for example, when $K = 0$, D has to be 0 and hence $P(K = 0, D = 1) = 0$), the positivity condition does not hold on S but there is a MRF where local structure is given as follows:

$$P(K = k|D = d) = \frac{P(K = k, D = d)}{P(D = d)} = \frac{\frac{exp\{-U(k,d)\}}{Z}}{h(p)^d(1 - h(p))^{1-d}}$$

and

$$P(D = d|K = k) = \frac{P(K = k, D = d)}{P(K = k)} = \frac{\frac{exp\{-U(k,d)\}}{Z}}{\binom{n}{k} p^k(1 - p)^{n-k}}$$

We now consider the Gibbs measure representation for the joint distribution $P(M = i, D = d) = P(i, d)$ on $I \times \{0, 1\} \equiv S^*$, where $I = \{i : s_i > 0\}$. Note that $P(i, d) > 0$ on S^* (it is a product space). The canonical Gibbs measure representation for (M, D) is given as follows.

From (2.2a),

$$\begin{aligned} P(M = i, D = 1) &= P(M = i)P(D = 1|M = i) \\ &= s_i \sum_{j=n-i+1}^{n} \binom{n}{j} p^j \overline{p}^{n-j} \end{aligned} \qquad (2.23)$$

and

$$\begin{aligned} P(M = i, D = 0) &= P(M = i) - P(M = i, D = 1) \\ &= s_i \sum_{j=0}^{n-i} \binom{n}{j} p^j \overline{p}^{n-j} \end{aligned} \qquad (2.24)$$

Let $U^*(i,d) = -(log\ s_i + d\ logC_i + (1-d)logD_i)$ and $Z^* = (1-p)^{-n}$,where

$$C_i = \sum_{j=n-i+1}^{n} \binom{n}{j} (\frac{p}{1-p})^j$$

and

$$D_i = \sum_{j=0}^{n-i} \binom{n}{j} (\frac{p}{1-p})^j.$$

Then,

$$P(M=i, D=d) = P(i,d) = \frac{exp\{-U^*(i,d)\}}{Z^*} \tag{2.25}$$

is the Gibbs measure representation where U^* is the canonical energy. The terms $-log\ s_i$ are capturing the ***complexity of the system structure*** since s_i is the system signature. The terms $-d\ log\ C_i - (1-d)logD_i$ are ***interaction terms*** where $\frac{C_i}{D_i}$ is the odds ratio of an i-out-of-n system working. This interaction is simply between D and these i out of n systems. The interaction terms indicate how the component reliabilities affect the system.

In summary and in terms of information theory, the uncertainty of structural reliability is composed of the uncertainty of structure (that is $-log\ s_i$) and the uncertainty of the component reliability (that is the interaction term).

Analogous to the case of (K, D), the local structure of the MRF is very simple since the field consists of two nodes, one for M and the other for D. Here

$$P(D=1|M=i) = \sum_{j=n-i+1}^{n} \binom{n}{j} p^j \overline{p}^{n-j}; P(D=0|M=i) = \sum_{j=0}^{n-i} \binom{n}{j} p^j \overline{p}^{n-j},$$

and

$$P(M=i|D=1) = \frac{s_i \sum_{j=n-i+1}^{n} \binom{n}{j} p^j \overline{p}^{n-j}}{h(p)}; P(M=i|D=0) = \frac{s_i \sum_{j=0}^{n-i} \binom{n}{j} p^j \overline{p}^{n-j}}{1-h(p)}.$$

3. Load-Sharing systems

To incorporate dependencies into the system reliability we will consider load-sharing systems. The earliest use of ***load-sharing*** systems was due to Daniels (1945) in the study of the probability distribution of the strength of a bundle of threads. It was also considered in Rosen's (1964) experimental work on fibrous composites. There he observed that, for a unidirectional fibrous composite under increasing tensile load, the load could not be transferred through the composite matrix material around fiber breaks for a vertical distance. He referred to this distance as the ***ineffective length***. This led to a series/parallel component system model for such composites based on the ineffective length where component/fiber information could be incorporated into the model.

Subsequent to Rosen's work, a series of authors, both material scientists and statisticians (e.g., Harlow et al, 1978, 1982), studied load-sharing systems where ***local load-sharing rules*** were used to incorporate the mechanical considerations into the model. More recent research and further references on load-sharing systems can be found in Harlow, Smith and Taylor (1983), Lee et al (1995) and Harlow (1997). Of particular interest is Harlow, Smith and Taylor (1983), who studied the extreme-value theory for ***monotone load-sharing*** systems.

The class of monotone load-sharing rules is a general class that contains a variety of rules such as equal, local and tapered load-sharing rules cited in the above references. Recently, Filliben (2005) and his colleagues at NIST used a load-sharing system with Weibull strengths for the grillage columns as part of the stochastic modeling of the grillage failure of the World Trade Center on 9/11. This was a component of the NIST study of the World Trade Center collapse.

In Section 3.1, we give some background for load-sharing parallel systems as well as a Gibbs Measure representation for such static systems. There an explicit formula is given to show how the potentials that define the representation are linear combinations of more primitive quantities, namely log odds ratios of the component strength distributions evaluated at loadings for subsets of the clique of the potential. In particular, when the component strength distributions are log-logistic, those log odds ratios are simple functions of the load-sharing rule. In Section 3.2, we consider the dynamic case where the parallel system of components is under an increasing load. There the Gibbs measure representation for the static case is extended and we prove a theorem and a series of lemmas showing that the dynamic systems have a Markovian structure.

3.1. Load-Sharing Systems: The Static Case

In Section 2.2, the Gibbs measure was given for general 0-1 systems where the local structure is given by the log odds ratios in (2.10). We now consider load-sharing parallel system where the system fails when all the components fail. For such systems, the related Markov random field is trivial in the sense that all the components (the nodes) are neighbors. But very interesting relationships can be obtained nonetheless. In particular, there are simple expressions for the local structure in terms of log odds of the component strength distributions and the load sharing rule.

The load-sharing and system are described as follows. The strength of a load-sharing system is based on the ***nominal load per component*** (the load per component), say x. We assume that there are n components in a system which are labeled by $N = \{1, 2, \ldots, n\}$. Let $M \subseteq N$ denote the set of working component in N, then the load at component $i \in M$ for a nominal load per component x is give by $\lambda_i(M)x$. These non-negative constants, $\lambda_i(M)$, define the load-sharing and the collection $\{\lambda_i(M) : i \in M, M \subseteq N\}$ is called a ***load-sharing rule***.

Different load-sharing rules have been used to help explain the stress concentrations in fibrous composite materials. Daniels (1945, for yarn) and Rosen (1964 for fiber glass composites) considered the model where the load is redistributed equally among unbroken fibers. This is referred as the ***equal load-sharing rule***. Another model introduced to account for local stress concentrations among unbroken fibers is the

local load-sharing rule (Harlow and Phoenix, 1978, 1982). The above rules are all special cases of a general class of rule referred to as ***monotone load-sharing rules***.

Following Harlow et al (1983), a load-sharing rule is referred to as a ***monotone load-sharing rule*** if

$$\lambda_i(L) \geq \lambda_i(M), \text{for all } i \in L, L \subseteq M \subseteq N \tag{3.1}$$

and

$$\sum_{i \in M} \lambda_i(M) > 0, M \subseteq N \tag{3.2}$$

A monotone load-sharing rule will be called ***strictly monotone*** if

$$\lambda_i(L) > \lambda_i(M), \text{for all } i \in L, L \subset M \subseteq N, \tag{3.3}$$

For load-sharing systems with component strength distributions $F_i, i = 1, \ldots, n$, $\sigma_i(A, s)$ defined in (2.10) is just

$$\sigma_i(A, s) = ln \frac{\overline{F}_i(\lambda_i(A)s)}{F_i(\lambda_i(A)s)} \tag{3.4}$$

Thus, σ_i is a simple log-odds ratio defined in terms of F_i.

Similar to the discussion in Section 2.2, We can define the Gibbs measure on the set of working components for load-sharing systems as

$$P_s(A) = \frac{1}{Z(s)} exp\{-U(\underline{\omega}(A), s)\} = \frac{1}{Z(s)} exp\{\sum_{K \subseteq A} V_K(\underline{\omega}(A), s)\}$$

where $P_s(A)$ is also referred to as the ***state probability*** of the system, U is the canonical energy and V is the nearest potential. By Theorem 2.2.2 and (3.4), the potentials now can be written as a function of the component distributions as

$$V_K(\underline{\omega}(K), s) = \frac{\sum_{A \subseteq K} \sum_{i \in A} (-1)^{|K-A|} ln \frac{\overline{F}_i(\lambda_i(A)s)}{F_i(\lambda_i(A)s)}}{|K|} \tag{3.5}$$

In classical thermodynamics, $U(\underline{\omega}(A), s)$ would denote the energy of state A of the system. Here it would be more meaningful to refer to $U(\underline{\omega}(A), s)$ as the ***ruin*** to the system when working components consist of those in the set of A. In section 2, we have obtained expressions for the ruin in terms of more primitive quantities, namely potentials and log odds ratios of the component reliabilities (see (2.12-13)). Note that log odds given in (3.4) is just the negative of the sum of hazard and reverse hazard functions of component i when the set of working components is A. The respective failure rates for these hazards might be interpreted as rates of ***damage*** and ***destruction***, respectively, in the context of a load-sharing.

When the strength distributions are generalizations of the log-logistic distributions, the log odds ratios and, as such, the potentials are simple functions of the load-sharing rules and are particularly simple for the log-logistic distributions. Such formulas are useful in assessing the structural reliability of the system and play a part in determining the system reliability, as we shall see in the remainder of this section, Section 3.2 and Section 4.

The simplification in the formulas is due in part to the following lemma.

Lemma 3.1.1 If $|K| > 1$, then $\sum_{A \subseteq K} \sum_{i \in A} (-1)^{|K-A|} = 0$.

Proof:

$$\begin{aligned}\sum_{A\subseteq K}\sum_{i\in A}(-1)^{|K-A|} &= \sum_{j=1}^{|K|}\binom{|K|}{j}j(-1)^{|K|-j} \qquad (j=|A|)\\ &= \sum_{j=1}^{|K|}|K|\binom{|K|-1}{j-1}(-1)^{(|K|-1)-(j-1)}\\ &= |K|\sum_{l=0}^{|K|-1}\binom{|K|-1}{l}(-1)^{(|K|-1)-l} \qquad (l=j-1)\\ &= |K|(1-1)^{|K|-1}\\ &= |K|\cdot 0^{|K|-1}=0 \text{ for } |k|>1. \qquad \square\end{aligned}$$

Suppose

$$F_i(x)=\frac{1}{1+(g(x)/\alpha_i)^{-\beta_i}}, x>0, \alpha_i>0, \beta_i>0, \tag{3.6}$$

is the strength distribution of component i. Thus, by (3.4),

$$\begin{aligned}\sigma_i(A,s) &= ln\frac{\overline{F}_i(\lambda_i(A)s)}{F_i(\lambda_i(A)s)} = ln(\frac{g(\lambda_i(A)s)}{\alpha_i})^{-\beta_i}\\ &= -\beta_i(ln\ g(\lambda_i(A)s)-ln\ \alpha_i)\end{aligned} \tag{3.7}$$

From Theorem 2.2.2 to (3.7), we get the following simple formulas for the potentials:

$$V_{\{i\}}(\underline{\omega}(\{i\}),s)=\sigma_i(\{i\},s)=-\beta_i(ln\ g(\lambda_i(\{i\})s)-ln\ \alpha_i)$$

and when $K\neq\{i\}$,

$$\begin{aligned}V_K(\underline{\omega}(K),s) &= \frac{\sum_{A\subseteq K}\sum_{i\in A}(-1)^{|K-A|}\sigma_i(A,s)}{|K|}\\ &= \frac{\sum_{A\subseteq K}\sum_{i\in A}(-1)^{|K-A|}(-\beta_i)ln\ g(\lambda_i(A)s)}{|K|}+\frac{\sum_{A\subseteq K}\sum_{i\in A}(-1)^{|K-A|}\beta_i ln\ \alpha_i}{|K|}\end{aligned} \tag{3.8}$$

When $\alpha_i\equiv\alpha$ and $\beta_i\equiv\beta$, the second term in (3.8) vanishes because of Lemma 3.1.1 and the potential does not depend on the scale parameter.

Formula (3.8) is even simpler for the special case of $g(x) = x$, where

$$\sigma_i(A, s) = -\beta_i(ln\ \lambda_i(A) + ln\ s - ln\ \alpha_i)$$

and formulas for the potentials become:

$$V_{\{i\}}(\underline{\omega}(\{i\}), s) = \sigma_i(\{i\}, s) = -\beta_i(ln\ \lambda_i(\{i\}) + ln\ s - ln\ \alpha_i)$$

and when $K \neq \{i\}$,

$$\begin{aligned} V_K(\underline{\omega}(K), s) \quad = \quad & \frac{\sum_{A\subseteq K}\sum_{i\in A}(-1)^{|K-A|}(-\beta_i)ln\ \lambda_i(A)}{|K|} \\ & + \frac{ln\ s\sum_{A\subseteq K}\sum_{i\in A}(-1)^{|K-A|}(-\beta_i)}{|K|} + \frac{\sum_{A\subseteq K}\sum_{i\in A}(-1)^{|K-A|}\beta_i ln\alpha_i}{|K|} \end{aligned} \tag{3.9}$$

In this case, when $\beta_i \equiv \beta$, the second term in (3.9) vanishes and it does not depend on s, while if both $\beta_i \equiv \beta$ and $\alpha_i \equiv \alpha$ (the so-called case of iid log-logistic strengths), only the first term does not vanish and it only depends on the load-sharing rule and β.

The constraint of identical scale parameters occurs in the context of inhomogeneous bundle of fibers with log-logistic strengths. Here the shape parameter is proportional to the cross-sectional area of the fiber. Under the assumption that the homogeneous sets of fibers with the same cross-sectional area are independent (thermal equilibrium), Gleaton and Lynch (2004) showed that for the max entropy principle to hold for the system and the subsystem of homogeneous fibers all the scale parameters must be the same.

3.2. The Markovian Structure for the Dynamic Case

When a system of components is subjected to an increasing load, the functioning components in that system share the load according to the load-sharing rule for the system. As the load is increased, components fail if their strengths are less than the load that is shared. Following Durham and Lynch (2000), there are two types of failures for an individual component. When a component fails directly due to the increasing load, it is referred as ***Phase I failure***. A Phase I failure can initiate a series of additional instantaneous failures (referred as ***Phase II failures***) as further load is transferred. The load that is transferred is also assumed to follow the system's load-sharing rule. If the system has not failed under this ***Phase I/II cycle***, the load is increasing and the system eventually undergoes another cycle. This continues until a collection of components fail that cause system failure.

To describe the load-sharing rule and the whole procedure of the system's failure, we introduce the notation of a ***breaking pattern***. The encoding notation for a breaking pattern is a listing in the form

$$p \equiv p_1 p_2 \dots p_f$$

where

$$p_m \equiv i_m(A_{m1}(A_{m2}(\dots(A_{mk_m})\dots))), m = 1, \dots, f \tag{3.10}$$

Components $i_1, \dots, i_f$ in the listing are not enclosed in parentheses and indicate components that undergo a Phase I failure. The index f is the number of Phase I/II cycles. Note that an isolated Phase I failure may initiate no Phase II failures but is still referred to as a Phase I/II cycle. The parenthetically nested collection of components between the Phase I failures indicate the Phase II failures they initiate. The number of right parentheses indicates the number of ***Phase II groups*** of components within a Phase I/II cycle. The groupings indicated by the preceding group's failure. For example, $i_1(A_{11}(A_{12}))i_2$ indicates that after component i_1 fails due to load increased, components in group A_{11} immediately fail due to load transfer because of the failure of i_1 but all other functioning components are able to accept their load. The failure of all components in A_{11} immediately causes group A_{12} to fail but all other functioning components are able to accept the redistributed load after A_{12} failed. The i_2 indicates the end of this Phase I/II cycle and that the load on the system needs to increase to initiate further component failures. The breaking pattern in (3.10) consists of f failure cycles and we refer p_m to as the m^{th} ***failure cycle***.

Let $O_m \equiv \cup_{v=1}^{k_m}\{\{i_m\} \cup A_{mv}\}$ be the set of components that failed in the m^{th} failure cycle. Note that p_f denotes the breaking pattern in the final failure cycle and O_f denotes the components that failed in the final failure cycle. The quantities p_f and O_f will be fundamental when we focus on the study on the system strength.

The Gibbs measure $P_s(A)$ and its associated MRF may be viewed as a static representation for the monotone load-sharing system. We will now give a dynamic representation. For arbitrary components lifetime/strength distributions, the component failures for a monotone load-sharing system under increasing load can be totally described as a Markovian structure. It consists of a series of isolated component failures (Phase I failures) followed by "instantaneous" failures (Phase II failures) caused by load transfer due to a Phase I failure. The set of Phase II failures in a given Phase I/II cycle is determined by a Markov random field. The Gibbs measure for the field can be obtained explicitly (similar to $P_s(A)$) where the potentials in its representation just involve log odds ratios of the component life distributions and the load-sharing rule.

Formally, under increasing load the system incurs component failures as follows:

- The system incurs a series of J Phase I failures due to load-sharing at nominal loads per components $S_1 < S_2 < \ldots < S_J = S$ where S is the system strength.

- The set of working components just prior to breaking stress S_i is A_i. Note that $A_1 = N = \{1, \ldots, n\}$. Let $X_{U(i)}$ denote the component strength of the component that fails at S_i. Then $X_{U(i)} = \lambda_{U(i)}(A_i)S_i$.

- When component $U(i)$ fails, sudden load transfer causes a series of component failures. The set of working components after these failures is A_{i+1}.

Assume that at the very beginning, the strengths of components, $X_i, i \in N$ are independent and follow the initial distribution $F_i, i \in N$ respectively. There are two interesting properties in the failure process. One is that prior to any failure cycle, the strengths of the current working components are ***conditionally independent (c.i.)*** given the previous Phase I breaking stress and the sets of working components. The second property is that there is a Markovian structure involved in this failure process such that what happens in the next failure cycle depends only on the current failure cycle. We will see that these two properties are just natural results from the proof of the "Basic Lemma" below.

Basic Lemma Prior to the Phase I failure of the m^{th} failure cycle, the strengths of the working components, $X_i, i \in A_m$, are **c.i.** given $S_1, \ldots, S_{m-1},\ A_1, \ldots, A_m$ with respective survival distribution

$$\overline{F}_i(x; A_m, S_{m-1}) \equiv \frac{\overline{F}_i(x)}{\overline{F}_i(\lambda_i(A_m)S_{m-1})},\ x > \lambda_i(A_m)S_{m-1}; m = 1, \ldots, J; S_0 \equiv 0.$$

and $S_m = \min\limits_{i \in A_m} \frac{X_i}{\lambda_i(A_m)}$.

Proof: We prove this lemma by mathematical induction. We use capital letters S_i, A_i and $U(i)$ to denote the random variables/sets and the lower case letters s_i, a_i and $u(i)$ to denote specific values of those random variables/sets.

Step 1 m=1.

$A_m = A_1 = N$. $X_i, i \in N$ are independent and follow the initial distribution $F_i, i \in N$, respectively, and

$$\overline{F}_i(x) = \frac{\overline{F}_i(x))}{\overline{F}_i(0)} = \frac{\overline{F}_i(x))}{\overline{F}_i(\lambda_i(A_1)S_0)} = \overline{F}_i(x; A_1, S_0),\ i \in N.$$

Thus, the statement in Basic Lemma holds for $m = 1$.

Step 2 Assume the Basic Lemma holds for $m = l$ for some $l : 1 \leq l \leq J - 1$.

Then $X_i, i \in A_l$ are **c.i.** given $S_1, \ldots, S_{l-1},\ A_1, \ldots, A_l$ with respective survival distribution

$$\overline{F}_i(x; A_l, S_{l-1}) = \frac{\overline{F}_i(x)}{\overline{F}_i(\lambda_i(A_l)S_{l-1})}, x > \lambda_i(A_l)S_{l-1}. \tag{3.11}$$

We want to show that $X_i, i \in A_{l+1}$ are **c.i.** given $S_1, \ldots, S_l$, $A_1, \ldots, A_{l+1}$ with respective survival distribution

$$\overline{F}_i(x; A_{l+1}, S_l) = \frac{\overline{F}_i(x)}{\overline{F}_i(\lambda_i(A_{l+1})S_l)}, x > \lambda_i(A_{l+1})S_l. \tag{3.12}$$

This can be done because of the following lemmas.

Lemma 3.2.1 $S_l|A_1, \ldots, A_l, S_1, \ldots, S_{l-1} \stackrel{d}{=} S_l|A_l, S_{l-1}$.

Proof: Under monotone load-sharing, $S_l = \min_{i \in A_l} \frac{X_i}{\lambda_i(A_l)}$. Thus,

$$\begin{aligned}
\overline{F}_{S_l|S_1,\ldots,S_{l-1},A_1,\ldots,A_l}(s|s_1, \ldots, s_{l-1}, a_1, \ldots, a_l) &= P(S_l > s|s_1, \ldots, s_{l-1}, a_1, \ldots, a_l) \\
&= P(X_i > \lambda_i(a_l)s, i \in a_l|s_1, \ldots, s_{l-1}, a_1, \ldots, a_l) \\
&= \prod_{i \in A_l} \overline{F}_i(\lambda_i(a_l)s; a_l, s_{l-1}) \qquad \text{(by (3.11))} \\
&= \prod_{i \in A_l} \frac{\overline{F}_i(\lambda_i(a_l)s)}{\overline{F}_i(\lambda_i(a_l)s_{l-1})}
\end{aligned} \tag{3.13}$$

which depends only on a_l, s_{l-1}. □

Lemma 3.2.2 $U(l)|A_1, \ldots, A_l, S_1, \ldots, S_{l-1}, S_l \stackrel{d}{=} U(l)|A_l, S_l$.

Proof:

$$
\begin{aligned}
& P(U(l)=u|s_1,\ldots,s_{l-1},s_l,a_1,\ldots,a_l) \\
= \; & f_{U(l)|S_1,\ldots,S_{l-1},S_l,A_1,\ldots,A_l}(u|s_1,\ldots,s_{l-1},s_l,a_1,\ldots,a_l) \\
= \; & \frac{f_{U(l)|S_1,\ldots,S_{l-1},A_1,\ldots,A_l}(u|s_1,\ldots,s_{l-1},a_1,\ldots,a_l)}{f_{S_l|S_1,\ldots,S_{l-1},A_1,\ldots,A_l}(s_l|s_1,\ldots,s_{l-1},a_1,\ldots,a_l)} \\
= \; & \frac{f_u(\lambda_u(a_l)s_l;a_l,s_{l-1}) \prod\limits_{k\neq u,k\in a_l} \overline{F}_k(\lambda_k(a_l)s_l;a_l,s_{l-1})}{\sum_{i\in a_l} f_i(\lambda_i(a_l)s_l;a_l,s_{l-1}) \prod\limits_{k\neq u,k\in a_l} \overline{F}_k(\lambda_k(a_l)s_l;a_l,s_{l-1})} \quad \text{(by (3.11), (3.12))} \quad (*) \\
= \; & \frac{\frac{f_u(\lambda_u(a_l)s_l)}{\overline{F}_u(\lambda_u(a_l)s_{l-1})} \prod\limits_{k\neq u,k\in a_l} \frac{\overline{F}_k(\lambda_k(a_l)s_l)}{\overline{F}_k(\lambda_k(a_l)s_{l-1})}}{\sum_{i\in a_l}\{\frac{f_i(\lambda_i(a_l)s_l)}{\overline{F}_i(\lambda_i(a_l)s_{l-1})} \prod\limits_{k\neq i,k\in a_l} \frac{\overline{F}_k(\lambda_k(a_l)s_l)}{\overline{F}_k(\lambda_k(a_l)s_{l-1})}\}} \quad \text{(by (3.13))} \\
= \; & \frac{f_u(\lambda_u(a_l)s_l) \prod\limits_{k\neq u,k\in a_l} \overline{F}_k(\lambda_k(a_l)s_l)}{\sum_{i\in a_l}\{f_i(\lambda_i(a_l)s_l) \prod\limits_{k\neq i,k\in a_l} \overline{F}_k(\lambda_k(a_l)s_l)\}} \qquad (3.14)
\end{aligned}
$$

The lemma follows since the last term in identity (3.14) only depends on s_l and a_l. □

Lemma 3.2.3 $X_i, i \in A_l - U(l)$ are **c.i.** and

$$X_i, i\in A_l-U(l)|A_1,\ldots,A_l,S_1,\ldots,S_l,U(l) \overset{d}{=} X_i, i\in A_l-U(l)|A_l,S_l,U(l).$$

Proof: Let $B_l \equiv A_l - U(l)$ and $\underline{X}_{B_l}$ be the random vector of strengths of components in B_l.

$$
\begin{aligned}
& f_{\underline{X}_{B_l}|S_1,\ldots,S_{l-1},S_l,A_1,\ldots,A_l,U(l)}(\underline{x}_{b_l}|s_1,\ldots,s_{l-1},s_l,a_1,\ldots,a_l,u(l)) \\
= & \frac{f_{\underline{X}_{B_l},U(l),S_l|S_1,\ldots,S_{l-1},A_1,\ldots,A_l}(\underline{x}_{b_l},u(l),s_l|s_1,\ldots,s_{l-1},a_1,\ldots,a_l)}{f_{U(l),S_l|S_1,\ldots,S_{l-1},A_1,\ldots,A_l}(u(l),s_l|s_1,\ldots,s_{l-1},a_1,\ldots,a_l)} \\
= & \frac{f_{u(l)}(\lambda_{u(l)}(a_l)s_l;a_l,s_{l-1})\prod_{k\in b_l} f_k(x_k;a_l,s_{l-1})}{f_{u(l)}(\lambda_{u(l)}(a_l)s_l;a_l,s_{l-1})\prod_{k\in b_l}\overline{F}_k(\lambda_k(a_l)s_l;a_l,s_{l-1})} \quad \text{(by (3.11) and (*))} \\
= & \prod_{k\in b_l}\frac{f_k(x_k;a_l,s_{l-1})}{\overline{F}_k(\lambda_k(a_l)s_l;a_l,s_{l-1})} \\
= & \prod_{k\in a_l-u(l)}\frac{f_k(x_k)}{\overline{F}_k(\lambda_k(a_l)s_l)} \equiv \prod_{k\in a_l-u(l)} f_k^*(x_k;s_l,a_l)
\end{aligned} \tag{3.15}
$$

Thus, Lemma 3.2.3 is true since the joint conditional density of $\underline{X}_{B_l}$ can be written as a product of conditional densities which depend only on s_l, a_l, and $u(l)$. □

Lemma 3.2.4 $A_{l+1}|A_1,\ldots,A_l,S_1,\ldots,S_l,U(l) \overset{d}{=} A_{l+1}|A_l,S_l,U(l)$.

Proof: we only need to show that

$$
P(A_{l+1}=al+1|s_1,\ldots,s_l,a_1,\ldots,a_l,U(l)=u(l)) = P(A_{l+1}=a_{l+1}|s_l,a_l,U(l)=u(l)) \tag{3.16}
$$

For the purpose of writing convenience in proving (3.16), we introduce a new notation as follows. For the notation of breaking pattern given in (3.10), the segment $(A_{11}(A_{22}(\ldots(A_{lk_l})\ldots)))$ gives the breaking pattern of Phase II failures in the m^{th} failure cycle. Consider the collection of sets $\{A_{11}, A12, \ldots, A_{lk_l}\}$. It defines a partition of the set $A_l - U(l) - A_{l+1}$. Let $\Pi(A_l, A_{l+1}, U(l))$ denote the collection of such partitions, then define $C \equiv \{C_1, C_2, C_3\}$, where

$$
C_1 \equiv \{X_i \le \lambda_i(A_l - U(l))S_l, i \in a_{l1}\};
$$

$$C_2 \equiv \{\lambda_i(A_l - U(l) - \bigcup_{r=0}^{v-2} a_{lr})S_l < X_i \leq \lambda_i(A_l - U(l) - \bigcup_{r=0}^{v-1} a_{lr})S_l, i \in a_{lv}, v = 2, \ldots, k_{l-1}\};$$

$$C_3 \equiv \{\lambda_i(a_{l+1})S_l < X_i, i \in a_{l+1}$$

with $a_{l0} \equiv \emptyset$, we have

$$\begin{aligned}
& P(A_{l+1} = a_{l+1} | s_1, \ldots, s_l, a_1, \ldots, a_l, U(l) = u(l))) \\
= & \sum_{\{a_{l1}, \ldots, a_{lkl}\} \in \Pi(a_l, a_{l+1}, u(l))} P(A_{l+1} = a_{l+1}, A_{lj} = a_{lj}, j = 1, \ldots, k_l | s_i, a_i, i = 1, \ldots, l; U(l) = u(l))) \\
= & \sum_{\{a_{l1}, \ldots, a_{lkl}\} \in \Pi(a_l, a_{l+1}, u(l))} P(A_{l+1} = a_{l+1}, C | s_i, a_i, i = 1, \ldots, l; U(l) = u(l))) \\
= & \sum_{\{a_{l1}, \ldots, a_{lkl}\} \in \Pi(a_l, a_{l+1}, u(l))} \{ \prod_{i \in a_{l1}} F^*(\lambda_i(a_l - u(l))s_l; s_l, a_l) \\
& \times \prod_{i \in a_{l2}} [F^*(\lambda_i(a_l - u(l) - a_{l1})s_l; s_l, a_l) - F^*(\lambda_i(a_l - u(l))s_l; s_l, a_l)] \\
& \times \ldots \\
& \times \prod_{i \in a_{l+1}} \overline{F}^*(\lambda_l(a_{l+1})s_l; s_l, a_l)\} \qquad \text{(by Lemma (3.2.3))} \qquad (3.17)
\end{aligned}$$

By Lemma 3.2.3, for an arbitrary partition $\pi = \{a_{l1}, \ldots, a_{lk_l}\} \in \Pi(a_l, a_{l+1}, u(l))$, the $F^{*'}$s in (3.17) are functions only through $s_l, a_l, u(l)$ and π . Thus,

$$\begin{aligned}
& \sum_{\{a_{l1}, \ldots, a_{lkl}\} \in \Pi(a_l, a_{l+1}, u(l))} P(A_{l+1} = a_{l+1}, A_{lj} = a_{lj}, j = 1, \ldots, kl | s_i, a_i, i = 1, \ldots, l; U(l) = u(l))) \\
= & \sum_{\{a_{l1}, \ldots, a_{lkl}\} \in \Pi(a_l, a_{l+1}, u(l))} P(A_{l+1} = a_{l+1}, A_{lj} = a_{lj}, j = 1, \ldots, kl | s_l, a_l; U(l) = u(l))) \\
= & P(A_{l+1} = a_{a+1} | s_l, a_l, U(l) = u(l)),
\end{aligned}$$

providing (3.16). Therefore,

$$\begin{aligned}
& P(A_{l+1} = a_{l+1} | s_1, \ldots, s_l, a_1, \ldots, a_l) \\
= & \sum_{u \in a_l - a_{l+1}} \{P(A_{l+1} = a_{l+1} | s_1, \ldots, s_l, a_1, \ldots, a_l, U(l) = u) \cdot P(U(l) = u | s_1, \ldots, s_l, a_1, \ldots, a_l)\} \\
= & \sum_{u \in a_l - a_{l+1}} \{P(A_{l+1} = a_{l+1} | s_1, \ldots, s_l, a_1, \ldots, a_l, U(l) = u) \cdot P(U(l) = u | s_l, a_l)\} \\
= & P(A_{l+1} = a_{l+1} | s_l, a_l) \qquad \square
\end{aligned}$$

Remark: from (3.17), we see that

$$\begin{aligned} &P((\underline{X}_{A_{l1}},\ldots,\underline{X}_{A_{lk_l}},\underline{X}_{A_{l+1}})\in\{\underline{x}=(\underline{x}_{A_{l1}},\ldots,\underline{x}_{A_{lk_l}},\underline{x}_{A_{l+1}}):\underline{x}\text{ consistent with }\pi\} \\ &\quad |s_i,a_i,i\le l,u(l),\pi) \\ &\quad =\prod_{j=1}^{k_l}\{\prod_{i\in A_{ij}}P(X_i\in\{x_i:x_i\text{ consistent with }\pi\}|s_i,a_i,i\le l,u(l)\pi)\} \\ &\quad \times\prod_{i\in a_{l+1}}P(X_i\in\{x_i:x_i\text{ consistent with }\pi\}|s_i,a_i,i\le l,u(l)\pi)\} \end{aligned}$$

From this, we observed the following facts,

(a) $X_i, i \in A_l - U(l)$ are conditionally independent given $S_1, \ldots, S_l, A_1, \ldots, A_l, U(l)$ and the partition $\pi \in \Pi(A_l, A_{l+1}, U(l))$, and the dependency is only through a smaller set $\{S_l, A_l, U(l), \pi \in \Pi(A_l, A_{l+1}, U(l))\}$.

(b) Given a realization $\{S_l = s_l, A_l = a_l, U(l) = u(l), \pi \in \Pi(A_l, A_{l+1}, U(l))\}$, $X_i, i \in a_{l+1}$ has the survival distribution

$$\overline{G}_i(x; s_l, a_l, u(l), \pi) = \frac{\overline{F}_i^*(x; s_l, a_l)}{\overline{F}_i^*(\lambda_i(a_{l+1})s_l; s_l, a_l)} \tag{3.18}$$

where $f_i^*(x; s_l, a_l) = \frac{f_i(x)}{\overline{F}_i(\lambda_i(a_l)s_l)}$.

Lemma 3.2.5 $X_i, i \in A_{l+1}|A_1, \ldots, A_{l+1}S_1, \ldots, S_l \sim \overline{F}_i(x; A_{l+1}, S_l)$.

Proof: By (3.18), the conditional distribution of $X_i, i \in a_{l+1}|s_1, \ldots, s_l, a_1, \ldots, a_l, u(l), \pi$ is

$$\begin{aligned}
\overline{G}_i(x; s_l, a_l, u(l), \pi) &= \frac{\overline{F}_i^*(x; s_l, a_l)}{\overline{F}_i^*(\lambda_i(a_{l+1})s_l; s_l, a_l)} \\
&= \frac{\overline{F}_i(x)/\overline{F}_i(\lambda_i(A_l)s_l)}{\overline{F}_i(\lambda_i(a_{l+1})s_l)/\overline{F}_i(\lambda_i(A_l)s_l)} \\
&= \frac{\overline{F}_i(x)}{\overline{F}_i(\lambda_i(a_{l+1})s_l))} \\
&= \overline{F}_i(x; a_{l+1}, s_l)
\end{aligned}$$

Since $\overline{F}_i(x; A_{l+1}, S_l)$ is a function only through A_{l+1} and S_l, we have

$$X_i, i \in A_{l+1}|S_1, \ldots, S_l, A_1, \ldots, A_l, U(l), \pi \in \Pi(A_l, A_{l+1}, U(l)) \stackrel{d}{=} X_i, i \in A_{l+1}|S_l, A_{l+1}$$

Note that

$$\sigma(\{S_l, A_{l+1}\}) \subseteq \sigma(\{S_1, \ldots, S_l, A_1, \ldots, A_{l+1}\}) \subseteq \sigma(\{S_i, A_i, i \leq l, U(l), \pi \in \Pi(A_l, A_{l+1}, U(l))\}),$$

therefore,

$$X_i, i \in A_{l+1}|S_1, \ldots, S_l, A_1, \ldots, A_{l+1} \stackrel{d}{=} X_i, i \in A_{l+1}|S_l, A_{l+1}$$

and hence the survival distribution of $X_i, i \in A_{l+1}|S_1, \ldots, S_l, A_1, \ldots, A_{l+1}$ is $\overline{F}_i(x; A + l + 1, S_l)$, finishing the proof of Lemma 3.2.5 as well as the proof of step 2. □

Therefore, by step 1 and 2, the Basic Lemma holds for $m = 1, \ldots, J$. □

The next theorem details the Markovian structure of the systems as we follow it through its Phase I/II cycles. Its proof is a consequence of the Basic Lemma, Lemma 3.2.1-3.2.5 and the discussion above. Note that we must now include A_m, $U(m)$ and S_m in the notation for σ_i and V_K to indicate the system of working components which survived the Phase I breaking stress S_{m-1}.

Theorem 3.2.1 Under monotone load-sharing, we have the following Markovian structure,

- $S_m|A_1,\ldots,A_m,S_1,\ldots,S_{m-1} \stackrel{d}{=} S_m|A_m,S_{m-1}$
- $U(m)|A_1,\ldots,A_m,S_1,\ldots,S_m,U(1),\ldots,U(m-1) \stackrel{d}{=} U(m)|A_m,S_m$
- $A_{m+1}|A_1,\ldots,A_m,S_1,\ldots,S_m,U(m) \stackrel{d}{=} A_{m+1}|S_m,A_m,U(m)$
- $\{(A_1,S_1,U(1)),(A_2,S_2,U(2)),\ldots\}$ is a Markov chain.
- $A_{m+1}|S_m,A_m,U(m)$ is a Markov random field on $A_m-\{U(m)\}$ with Gibbs measure

$$P_{S_m,A_m,U(m)}(A), A \subseteq A_m - \{U(m)\}. \tag{3.19}$$

Given S_m, A_m and $U(m)$, for $A \subseteq A_m - \{U(m)\}$,

$$P_{S_m,A_m,U(m)}(A) = \frac{exp\{\sum_{K\subseteq A} V_K(\underline{\omega}(A);A_m,S_m,U(m))\}}{Z(S_m,A_m,U(m))}, \tag{3.20}$$

and

$$\begin{aligned} \sigma_i(A;A_m,S_m,U(m)) &= \sum_{\{K\subseteq A: i\in K\}} V_K(\underline{\omega}(A);A_m,S_m,U(m)) \\ &= ln\frac{\overline{F}_i(\lambda_i(A)S_m;A_m,S_m)}{F_i(\lambda_i(A)S_m;A_m,S_m)}. \end{aligned} \tag{3.21}$$

4. Reliability Concepts for Load-Sharing Systems

In Section 3.1, we obtained a formula for the potentials in terms of the load-sharing rule and the log odds of the distribution of the component strengths for the case of a static load-sharing system. This formula is easier to interpret when the component strengths have log-logistic distributions or generalizations of the log-logistic distributions. Thus it can be used to assess the system structure reliability. For the dynamic case as described in Section 3.2, we will see that the formula for potentials is no longer that simple but consists of two terms where one term, though, is just the potential, $V_k(\underline{\omega}(K),S_m)$, from the static case. The other term, then, can be thought of as accounting for the additional ***complexity*** introduced due to the model being dynamic. The purpose of this section is to formalize such relationships and the concept of ***structural reliability***.

Consider the dynamic case and let $\Omega_i(x) \equiv F_i(x)/\overline{F}_i(x)$ be the ***odds of failure*** for component i. (Note that $\sigma_i(A,s) = -ln\ \Omega_i(\lambda_i(A)s)$.) By Theorem 2.2.2, the Basic Lemma and (3.21), given the stress level $s = S_m$, the current working set A_m and the next Phase I failure component $U(m)$, the potential of K is

$$
\begin{aligned}
V_K(\underline{\omega}(K), A_m, S_m, U(m)) &= \frac{\sum_{A\subseteq K}\sum_{i\in A}(-1)^{|K-A|}\sigma_i(A, A_m, S_m, U(m))}{|K|} \\
&= \frac{\sum_{A\subseteq K}\sum_{i\in A}(-1)^{|K-A|} ln \frac{\overline{F}_i(\lambda_i(A)S_m, A_m, S_m)}{F_i(\lambda_i(A)S_m; A_m, S_m)}}{|K|} \\
&= \frac{\sum_{A\subseteq K}\sum_{i\in A}(-1)^{|K-A|} ln \frac{\overline{F}_i(\lambda_i(A)S_m)}{\overline{F}_i(\lambda_i(A_m)S_m) - \overline{F}_i(\lambda_i(A)S_m)}}{|K|} \\
&= \frac{\sum_{A\subseteq K}\sum_{i\in A}(-1)^{|K-A|}\sigma_i(A, S_m)}{|K|} \\
&\quad - \frac{\sum_{A\subseteq K}\sum_{i\in A}(-1)^{|K-A|} ln(\frac{\overline{F}_i(\lambda_i(A_m)S_m)}{F_i(\lambda_i(A)S_m)} - (\Omega_i(\lambda_i(A)S_m)^{-1})}{|K|} \\
&= V_K(\underline{\omega}(K), S_m) - C(U(m), A_m, S_m) \qquad (4.1)
\end{aligned}
$$

where

$$
C(U(m), A_m, S_m) \equiv \frac{\sum_{A\subseteq K}\sum_{i\in A}(-1)^{|K-A|} ln(\frac{\overline{F}_i(\lambda_i(A_m)S_m)}{F_i(\lambda_i(A)S_m)} - (\Omega_i(\lambda_i(A)S_m)^{-1})}{|K|}
$$

is the term for the complexity of the dynamic system.

Thus, the relationship between the Gibbs measure representations for the static and dynamic case is clearly described as in (4.1) through the relationship between their potentials. In the complexity term, $\Omega_i(\lambda_i(A)S_m)$ is the odds of component failure under the static case and only $\frac{\overline{F}_i(\lambda_i(A_m)S_m)}{F_i(\lambda_i(A)S_m)}$ depends on the dynamic process. It shows how component information and information from static testing can be incorporated into in assessing system reliability as well as being useful in system design.

5. Examples

In this section, we consider equal load-sharing and local load-sharing for parallel systems of size $n = 4$ and $n = 5$ for iid component strength distributions F. We illustrate how to find the potentials and state probabilities for the static case through the following examples.

Example 5.1 Equal Load-Sharing

For equal load-sharing, $\lambda_i(A) = \frac{n}{|A|}$ where $A \subseteq N$ is the set of working components. Under this rule, by (3.5) the potential of K under stress s is

$$V_K(\underline{\omega}(K), s) = \frac{\sum_{j=1}^{|K|} \binom{|K|}{j} j(-1)^{|K|-j} ln \frac{\overline{F}(\frac{ns}{j})}{F(\frac{ns}{j})}}{|K|} \equiv V_n^{(|K|)}(s) \tag{5.1}$$

since the log odds ratios only depend on the size of the set of working components. From (5.1), when $n = 4$,

$$\begin{aligned}
V_4^{(1)}(s) &= ln\frac{\overline{F}(4s)}{F(4s)} \\
V_4^{(2)}(s) &= -ln\frac{\overline{F}(4s)}{F(4s)} + ln\frac{\overline{F}(2s)}{F(2s)} \\
V_4^{(3)}(s) &= ln\frac{\overline{F}(4s)}{F(4s)} - 2ln\frac{\overline{F}(2s)}{F(2s)} + ln\frac{\overline{F}(4s/3)}{F(4s/3)} \\
V_4^{(4)}(s) &= -ln\frac{\overline{F}(4s)}{F(4s)} + 3ln\frac{\overline{F}(2s)}{F(2s)} - 3ln\frac{\overline{F}(4s/3)}{F(4s/3)} + ln\frac{\overline{F}(s)}{F(s)}
\end{aligned} \tag{5.2}$$

Let $P_{s,n}^{(m)}$ denote the probability of the state when the number of working components is m, $m = 1, 2, 3, 4$ and the number of components in the system is n. Thus, by the second equality of (2.12),

$$
\begin{aligned}
P_{s,4}^{(1)} &= \frac{1}{Z_4(s)}\binom{4}{1}(\frac{\overline{F}(4s)}{F(4s)}) \\
P_{s,4}^{(2)} &= \frac{1}{Z_4(s)}\binom{4}{2}(\frac{\overline{F}(4s)}{F(4s)})(\frac{\overline{F}(2s)}{F(2s)}) \\
P_{s,4}^{(3)} &= \frac{1}{Z_4(s)}\binom{4}{3}(\frac{\overline{F}(4s)}{F(4s)})(\frac{\overline{F}(2s)}{F(2s)})(\frac{\overline{F}(4s/3)}{F(4s/3)}) \\
P_{s,4}^{(4)} &= \frac{1}{Z_4(s)}\binom{4}{4}(\frac{\overline{F}(4s)}{F(4s)})(\frac{\overline{F}(2s)}{F(2s)})(\frac{\overline{F}(4s/3)}{F(4s/3)})(\frac{\overline{F}(s)}{F(s)}) \qquad (5.3)
\end{aligned}
$$

where $Z_4(s) = P_{s,4}(\emptyset)^{-1} = (1 - (P_{s,4}^{(1)} + P_{s,4}^{(2)} + P_{s,4}^{(3)} + P_{s,4}^{(4)}))^{-1}$.

Similarly, the potentials and state probabilities for $n = 5$ are

$$
\begin{aligned}
V_5^{(1)}(s) &= ln\frac{\overline{F}(5s)}{F(5s)} \\
V_5^{(2)}(s) &= -ln\frac{\overline{F}(5s)}{F(5s)} + ln\frac{\overline{F}(5s/2)}{F(5s/2)} \\
V_5^{(3)}(s) &= ln\frac{\overline{F}(5s)}{F(5s)} - 2ln\frac{\overline{F}(5s/2)}{F(5s/2)} + ln\frac{\overline{F}(5s/3)}{F(5s/3)} \\
V_5^{(4)}(s) &= -ln\frac{\overline{F}(5s)}{F(5s)} + 3ln\frac{\overline{F}(5s/2)}{F(5s/2)} - 3ln\frac{\overline{F}(5s/3)}{F(5s/3)} + ln\frac{\overline{F}(5s/4)}{F(5s/4)} \\
V_5^{(5)}(s) &= ln\frac{\overline{F}(5s)}{F(5s)} - 4ln\frac{\overline{F}(5s/2)}{F(5s/2)} + 6ln\frac{\overline{F}(5s/3)}{F(5s/3)} - 4ln\frac{\overline{F}(5s/4)}{F(5s/4)} + +ln\frac{\overline{F}(s)}{F(s)} \qquad (5.4)
\end{aligned}
$$

and

$$
\begin{aligned}
P_{s,5}^{(1)} &= \frac{1}{Z_5(s)}\binom{5}{1}(\frac{\overline{F}(5s)}{F(5s)}) \\
P_{s,5}^{(2)} &= \frac{1}{Z_5(s)}\binom{5}{2}(\frac{\overline{F}(5s)}{F(5s)})(\frac{\overline{F}(5s/2)}{F(5s/2)}) \\
P_{s,5}^{(3)} &= \frac{1}{Z_5(s)}\binom{5}{3}(\frac{\overline{F}(5s)}{F(5s)})(\frac{\overline{F}(5s/2)}{F(5s/2)})(\frac{\overline{F}(5s/3)}{F(5s/3)}) \\
P_{s,5}^{(4)} &= \frac{1}{Z_5(s)}\binom{5}{4}(\frac{\overline{F}(5s)}{F(5s)})(\frac{\overline{F}(5s/2)}{F(5s/2)})(\frac{\overline{F}(5s/3)}{F(5s/3)})(\frac{\overline{F}(5s/4)}{F(5s/4)}) \\
P_{s,5}^{(5)} &= \frac{1}{Z_5(s)}\binom{5}{5}(\frac{\overline{F}(5s)}{F(5s)})(\frac{\overline{F}(5s/2)}{F(5s/2)})(\frac{\overline{F}(5s/3)}{F(5s/3)})(\frac{\overline{F}(5s/4)}{F(5s/4)})(\frac{\overline{F}(s)}{F(s)}) \qquad (5.5)
\end{aligned}
$$

where $Z_5(s) = P_{s,5}(\emptyset)^{-1} = (1 - (P_{s,5}^{(1)} + P_{s,5}^{(2)} + P_{s,5}^{(3)} + P_{s,5}^{(4)} - P_{s,5}^{(5)}))^{-1}$.

When the strengths distributions are log-logistic, from (5.2)-(5.5) the potentials and state probabilities for $n = 4$ and $n = 5$ are

$$
\begin{aligned}
V_4^{(1)}(s) &= -\beta(ln\ 4 + ln\ s - ln\ \alpha), & V_5^{(1)}(s) &= -\beta(ln\ 5 + ln\ s - ln\ \alpha) \\
V_4^{(2)}(s) &= \beta\ ln\ 2, & V_5^{(2)}(s) &= \beta\ ln\ 2 \\
V_4^{(3)}(s) &= -\beta\ ln\ \frac{4}{3}, & V_5^{(3)}(s) &= -\beta\ ln\ \frac{4}{3} \\
V_4^{(4)}(s) &= \beta\ ln\ \frac{32}{27}, & V_5^{(4)}(s) &= \beta\ ln\ \frac{32}{27} \\
& & V_5^{(5)}(s) &= -\beta\ ln\ \frac{4096}{3645}
\end{aligned}
\tag{5.6}
$$

and

$$
\begin{aligned}
P_{s,4}^{(1)} &= \frac{1}{Z_4(s)}\binom{4}{1}(\frac{4s}{\alpha})^{-\beta}, & P_{s,5}^{(1)} &= \frac{1}{Z_5(s)}\binom{5}{1}(\frac{5s}{\alpha})^{-\beta} \\
P_{s,4}^{(2)} &= \frac{1}{Z_4(s)}\binom{4}{2}(\frac{(4s)^2}{2!\ \alpha^2})^{-\beta}, & P_{s,5}^{(2)} &= \frac{1}{Z_5(s)}\binom{5}{2}(\frac{(5s)^2}{2!\ \alpha^2})^{-\beta} \\
P_{s,4}^{(3)} &= \frac{1}{Z_4(s)}\binom{4}{3}(\frac{(4s)^3}{3!\ \alpha^3})^{-\beta}, & P_{s,5}^{(3)} &= \frac{1}{Z_5(s)}\binom{5}{3}(\frac{(5s)^3}{3!\ \alpha^3})^{-\beta} \\
P_{s,4}^{(4)} &= \frac{1}{Z_4(s)}\binom{4}{4}(\frac{(4s)^4}{4!\ \alpha^4})^{-\beta}, & P_{s,5}^{(4)} &= \frac{1}{Z_5(s)}\binom{5}{4}(\frac{(5s)^4}{4!\ \alpha^4})^{-\beta} \\
& & P_{s,5}^{(5)} &= \frac{1}{Z_5(s)}\binom{5}{5}(\frac{(5s)^5}{5!\ \alpha^4})^{-\beta}
\end{aligned}
$$

where

$$Z_n(s) = P_{s,n}(\emptyset)^{-1} = (1 - \sum_{i=1}^{n} P_{s,n}^{(i)})^{-1} \tag{5.7}$$

Example 5.2 Local Load-Sharing

In most of the cases, the equal load-sharing rule is not appropriate for modeling stress concentration. A more realistic load-sharing rule was studied by Harlow et al, (1978, 1982) for the chain of bundle model. There the load-sharing is defined as

$$\lambda_i(A) = 1 + r/2$$

where r is the number of failed components immediately adjacent to component i.

Here we illustrate how to find the Gibbs measure under this local load-sharing rule for a circular bundles of four and five components systems. Note that the load-sharing for this case is the same as that for the equal load-sharing model with $n = 4$ except when there are exactly three working components. For three working components, the load are $3s/2, 3s/2$ and s. Thus, the potentials for cliques of different sizes are

$$\begin{aligned}
V_4^{(1)}(s) &= ln\frac{\overline{F}(4s)}{F(4s)} \\
V_4^{(2)}(s) &= -ln\frac{\overline{F}(4s)}{F(4s)} + ln\frac{\overline{F}(2s)}{F(2s)} \\
V_4^{(3)}(s) &= ln\frac{\overline{F}(4s)}{F(4s)} - 2ln\frac{\overline{F}(2s)}{F(2s)} + \frac{2}{3}ln\frac{\overline{F}(3s/2)}{F(3s/2)} + \frac{1}{3}ln\frac{\overline{F}(s)}{F(s)} \\
V_4^{(4)}(s) &= -ln\frac{\overline{F}(4s)}{F(4s)} + 3ln\frac{\overline{F}(2s)}{F(2s)} - 2ln\frac{\overline{F}(3s/2)}{F(3s/2)}
\end{aligned}$$

and the Gibbs measure for state of components are

$$\begin{aligned}
P_{s,4}^{(1)} &= \frac{1}{Z_4(s)}\binom{4}{1}(\frac{\overline{F}(4s)}{F(4s)}) \\
P_{s,4}^{(2)} &= \frac{1}{Z_4(s)}\binom{4}{2}(\frac{\overline{F}(4s)}{F(4s)})(\frac{\overline{F}(2s)}{F(2s)}) \\
P_{s,4}^{(3)} &= \frac{1}{Z_4(s)}\binom{4}{3}(\frac{\overline{F}(4s)}{F(4s)})(\frac{\overline{F}(2s)}{F(2s)})(\frac{\overline{F}(3s/2)}{F(3s/2)})^{\frac{2}{3}}(\frac{\overline{F}(s)}{F(s)})^{\frac{1}{3}} \\
P_{s,4}^{(4)} &= \frac{1}{Z_4(s)}\binom{4}{4}(\frac{\overline{F}(4s)}{F(4s)})(\frac{\overline{F}(2s)}{F(2s)})(\frac{\overline{F}(3s/2)}{F(3s/2)})^{\frac{2}{3}}(\frac{\overline{F}(s)}{F(s)})^{\frac{4}{3}}
\end{aligned}$$

where $Z_4(s)$ is given by (5.7).

When the component strength distributions are iid log-logistic, we have

$$\begin{aligned}
V_4^{(1)}(s) &= -\beta(ln\ 4 + ln\ s - ln\ \alpha) \\
V_4^{(2)}(s) &= \beta\ ln\ 2 \\
V_4^{(3)}(s) &= -\beta \cdot \frac{2}{3}\ ln\frac{3}{2} \\
V_4^{(4)}(s) &= \beta\ ln\frac{9}{8}
\end{aligned} \tag{5.8}$$

and

$$
\begin{aligned}
P^{(1)}_{s,4} &= \frac{1}{Z_4(s)}\binom{4}{1}(4\frac{s}{\alpha})^{-\beta}\\
P^{(2)}_{s,4} &= \frac{1}{Z_4(s)}\binom{4}{2}(8(\frac{s}{\alpha})^2)^{-\beta}\\
P^{(3)}_{s,4} &= \frac{1}{Z_4(s)}\binom{4}{3}(8(\frac{3}{2})^{2/3}(\frac{s}{\alpha})^3)^{-\beta}\\
P^{(4)}_{s,4} &= \frac{1}{Z_4(s)}\binom{4}{4}(8(\frac{3}{2})^{2/3}(\frac{s}{\alpha})^4)^{-\beta}
\end{aligned}
$$

For $n = 5$, the local load-sharing rule is different from the equal load-sharing rule when there are exactly 3 ($|A| = 3$) or 4 ($|A| = 4$) working components . If $|A| = 4$, the load is distributed as $(s, s, 3s/2, 3s/2)$ while $|A| = 3$ leads to a more complicated situation. For $|A| = 3$, among the 10 combinations of 3 components out of 5, the load on the 3 working components are either $(s, 2s, 2s)$ or $(2s, 3s/2, 3s/2)$ with the same multiplicity of 5. Under this load-sharing rule the potentials of cliques of size 3 have two different values, denoted by $V_5^{(3,1)}$ and $V_5^{(3,2)}$ as given in the following formulas:

$$
\begin{aligned}
V_5^{(1)}(s) &= ln\frac{\overline{F}(5s)}{F(5s)}\\
V_5^{(2)}(s) &= -ln\frac{\overline{F}(5s)}{F(5s)} + ln\frac{\overline{F}(5s/2)}{F(5s/2)}\\
V_5^{(3,1)}(s) &= ln\frac{\overline{F}(5s)}{F(5s)} - 2ln\frac{\overline{F}(5s/2)}{F(5s/2)} + \frac{2}{3}ln\frac{\overline{F}(2s)}{F(2s)} + \frac{1}{3}ln\frac{\overline{F}(s)}{F(s)}\\
V_5^{(3,2)}(s) &= ln\frac{\overline{F}(5s)}{F(5s)} - 2ln\frac{\overline{F}(5s/2)}{F(5s/2)} + \frac{2}{3}ln\frac{\overline{F}(3s/2)}{F(3s/2)} + \frac{1}{3}ln\frac{\overline{F}(2s)}{F(2s)}\\
V_5^{(4)}(s) &= -ln\frac{\overline{F}(5s)}{F(5s)} + 3ln\frac{\overline{F}(5s/2)}{F(5s/2)} - \frac{3}{2}ln\frac{\overline{F}(2s)}{F(2s)} - \frac{1}{2}ln\frac{\overline{F}(3s/2)}{F(3s/2)}\\
V_5^{(5)}(s) &= ln\frac{\overline{F}(5s)}{F(5s)} - 4ln\frac{\overline{F}(5s/2)}{F(5s/2)} + 3ln\frac{\overline{F}(2s)}{F(2s)} \qquad (5.9)
\end{aligned}
$$

and the formulas for state probabilities are:

$$\begin{aligned}
P_{s,5}^{(1)} &= \frac{1}{Z_5(s)}\binom{5}{1}(\frac{\overline{F}(5s)}{F(5s)}) \\
P_{s,5}^{(2)} &= \frac{1}{Z_5(s)}\binom{5}{2}(\frac{\overline{F}(5s)}{F(5s)})(\frac{\overline{F}(5s/2)}{F(5s/2)}) \\
P_{s,5}^{(3)} &= \frac{1}{Z_5(s)}5(\frac{\overline{F}(5s)}{F(5s)})(\frac{\overline{F}(5s/2)}{F(5s/2)})[(\frac{\overline{F}(2s)}{F(2s)})^{\frac{2}{3}}(\frac{\overline{F}(s)}{F(s)})^{\frac{1}{3}} + (\frac{\overline{F}(2s)}{F(2s)})^{\frac{1}{3}}(\frac{\overline{F}(3s/2)}{F(3s/2)})^{\frac{2}{3}}] \\
P_{s,5}^{(4)} &= \frac{1}{Z_5(s)}\binom{5}{4}(\frac{\overline{F}(5s)}{F(5s)})(\frac{\overline{F}(5s/2)}{F(5s/2)})(\frac{\overline{F}(2s)}{F(2s)})^{\frac{1}{2}}(\frac{\overline{F}(3s/2)}{F(3s/2)})^{\frac{5}{6}}(\frac{\overline{F}(s)}{F(s)})^{\frac{2}{3}} \\
P_{s,5}^{(5)} &= \frac{1}{Z_5(s)}\binom{5}{5}(\frac{\overline{F}(5s)}{F(5s)})(\frac{\overline{F}(5s/2)}{F(5s/2)})(\frac{\overline{F}(2s)}{F(2s)})^{\frac{1}{2}}(\frac{\overline{F}(3s/2)}{F(3s/2)})^{\frac{5}{6}}(\frac{\overline{F}(s)}{F(s)})^{\frac{5}{3}} \qquad (5.10)
\end{aligned}$$

When the component strength distributions are log-logistic, from (5.9) and (5.10) we get

$$\begin{aligned}
V_5^{(1)}(s) &= -\beta(ln\ 5 + ln\ s - ln\ \alpha) \\
V_5^{(2)}(s) &= \beta\ ln\ 2 \\
V_5^{(3,1)}(s) &= \beta(ln\ 5 - \frac{8}{3}ln\ 2) \\
V_5^{(3,2)}(s) &= \beta(ln\ 5 - \frac{2}{3}ln\ 3 - \frac{5}{3}ln\ 2) \\
V_5^{(4)}(s) &= -\beta(2ln\ 5 - \frac{1}{2}ln\ 3 - 4ln\ 2) \\
V_5^{(5)}(s) &= \beta(3ln\ 5 - 7ln\ 2) \qquad (5.11)
\end{aligned}$$

and

$$\begin{aligned}
P_{s,5}^{(1)} &= \frac{1}{Z_5(s)}\binom{5}{1}(5\frac{s}{\alpha})^{-\beta} \\
P_{s,5}^{(2)} &= \frac{1}{Z_5(s)}\binom{5}{2}(\frac{5^2}{2}(\frac{s}{\alpha})^2)^{-\beta} \\
P_{s,5}^{(3)} &= \frac{1}{Z_5(s)}5((5^2 2^{-\frac{1}{3}}(\frac{s}{\alpha})^3)^{-\beta} + (5^2 3^{\frac{2}{3}} 2^{-\frac{4}{3}})^{-\beta}) \\
P_{s,5}^{(4)} &= \frac{1}{Z_5(s)}\binom{5}{4}(5^2 3^{\frac{5}{6}} 2^{-\frac{4}{3}}(\frac{s}{\alpha})^4)^{-\beta} \\
P_{s,5}^{(5)} &= \frac{1}{Z_5(s)}\binom{5}{5}(5^2 3^{\frac{5}{6}} 2^{-\frac{4}{3}}(\frac{s}{\alpha})^5)^{-\beta}
\end{aligned}$$

On the next few pages, some graphs are given that compare the potentials, state probabilities and system reliabilities under equal load-sharing and local load-sharing rule for the static case. We use log-logistic distribution with $\alpha = \beta = 1$ as the component strength distribution. This choice of α and β is a Pareto distribution, which is a very heavy tailed distribution. In Figures 5.1-4, $s = 0.1, 0.5, 1$, where $s = 1$ is just the median strength of the component when $\alpha = 1$.

Figures 5.1-2 are the graphs for the potentials. Note that, as indicated by (3.9), the potentials for cliques of size two or more do not depend on s. Figures 5.3-5.4 indicate how the probability distribution changes as the load increases. The greater the load, the more likely that only a small number of components survive. Comparing Figures 5.1 to 5.2, we see that there are only small difference between the potentials under the two load-sharing rule. In fact, as given in (5.6), (5.8) and (5.11), the potentials of cliques of size one and two are the same for both load-sharing rules. This similarity in potentials causes the similarity in their state probability distributions as shown in Figures 5.3-4.

Figure 5.5 compares the reliabilities under equal load sharing and local load-sharing when $n = 4$ and $n = 5$. The difference of the reliabilities under equal load-sharing and local load-sharing is hardly discernible for systems of the same size. In fact, a further calculation indicates that, the reliability for the local load-sharing system is always greater than that of the equal load-sharing system for $n = 4$ and $n = 5$. This fact might be counterintuitive but can happen with certain choices of the component strength distribution.

Figure 5.6-12 show how the potentials, state probabilities and system reliabilities change for equal load-sharing systems with $n = 4$ and $n = 5$ for various choices of α, β, and s. The behavior of local load-sharing systems is similar and not displayed.

As noted earlier, only the size 1 potentials depend on s. They are decreasing in s without limits. As seen in Figure 5.6, the size 1 potentials dominate the other potentials for extreme values of s. In that case, the load-sharing system can be approximated by a system of independent Bernoulli components where its parameter p is close to 0 for large s and close to 1 for small s (see Figure 5.7). Figure 5.8 and 5.11 show that the larger the α is, the greater the probability that more components are working and the stronger the system is. The effect of changing β is shown in Figure 5.9, 5.10 and 5.12 and depends on the ratio of s/α. If s/α is large, the system reliability is a decreasing function of β, and as β increases the probabilities move from states of larger working sets to those of smaller working sets and tend to cluster at the state of the empty set. When s/α is small, the system reliability is an increasing function of β and the state probabilities move the other way around.

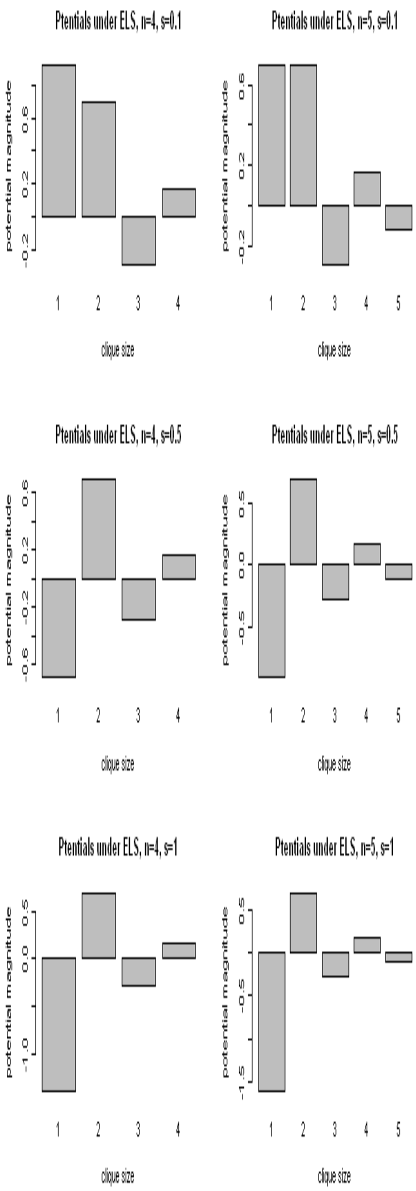

Figure 5.1: Potentials under Equal Load-Sharing (ELS) for log-logistic component strength distributions

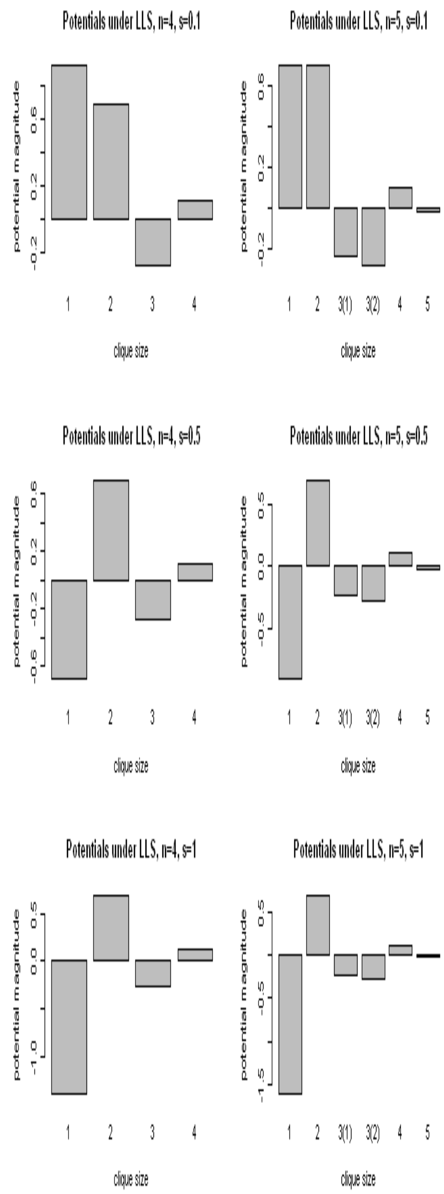

Figure 5.2: Potentials under Local Load-Sharing (LLS) for log-logistic component strength distributions

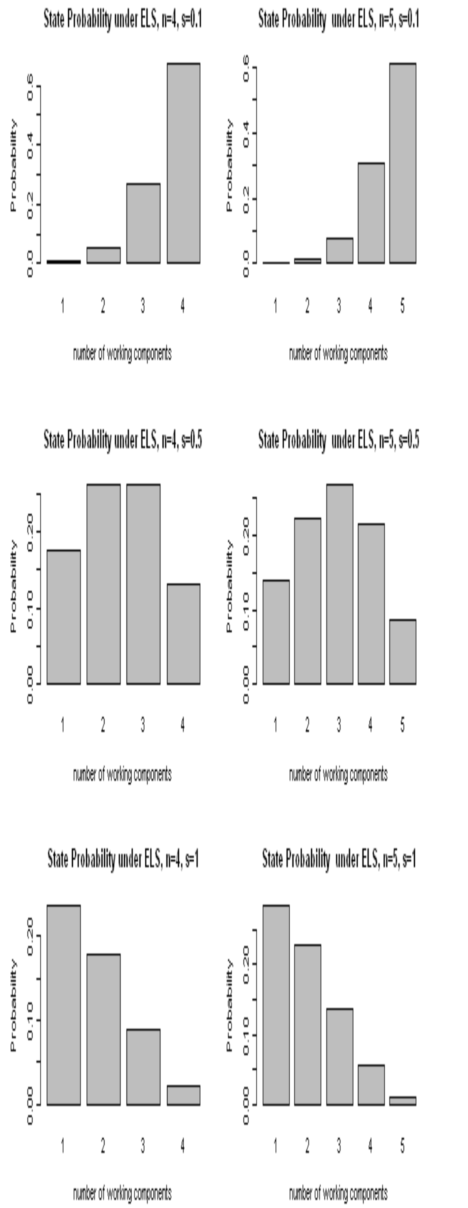

Figure 5.3: State probability under Equal Load-Sharing (ELS) for log-logistic component strength distributions

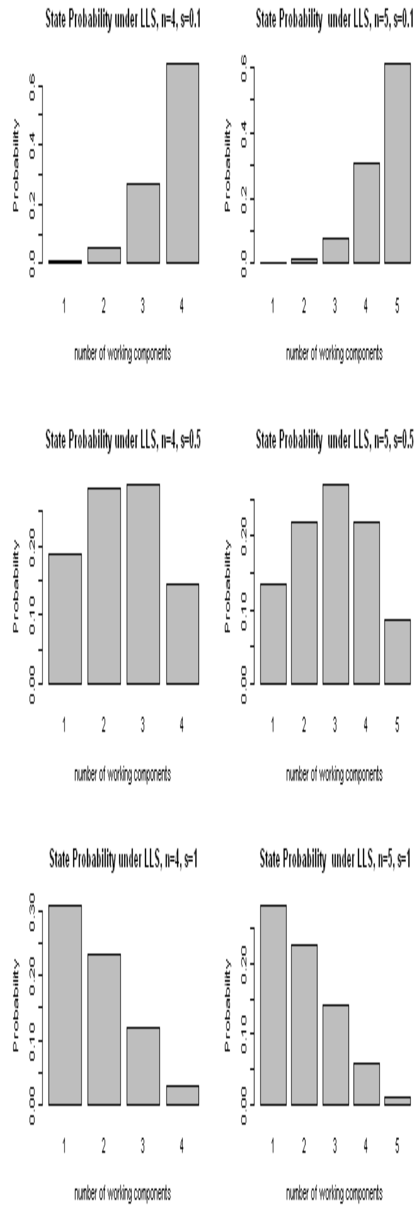

Figure 5.4: State probability under Local Load-Sharing (LLS) for log-logistic component strength distributions

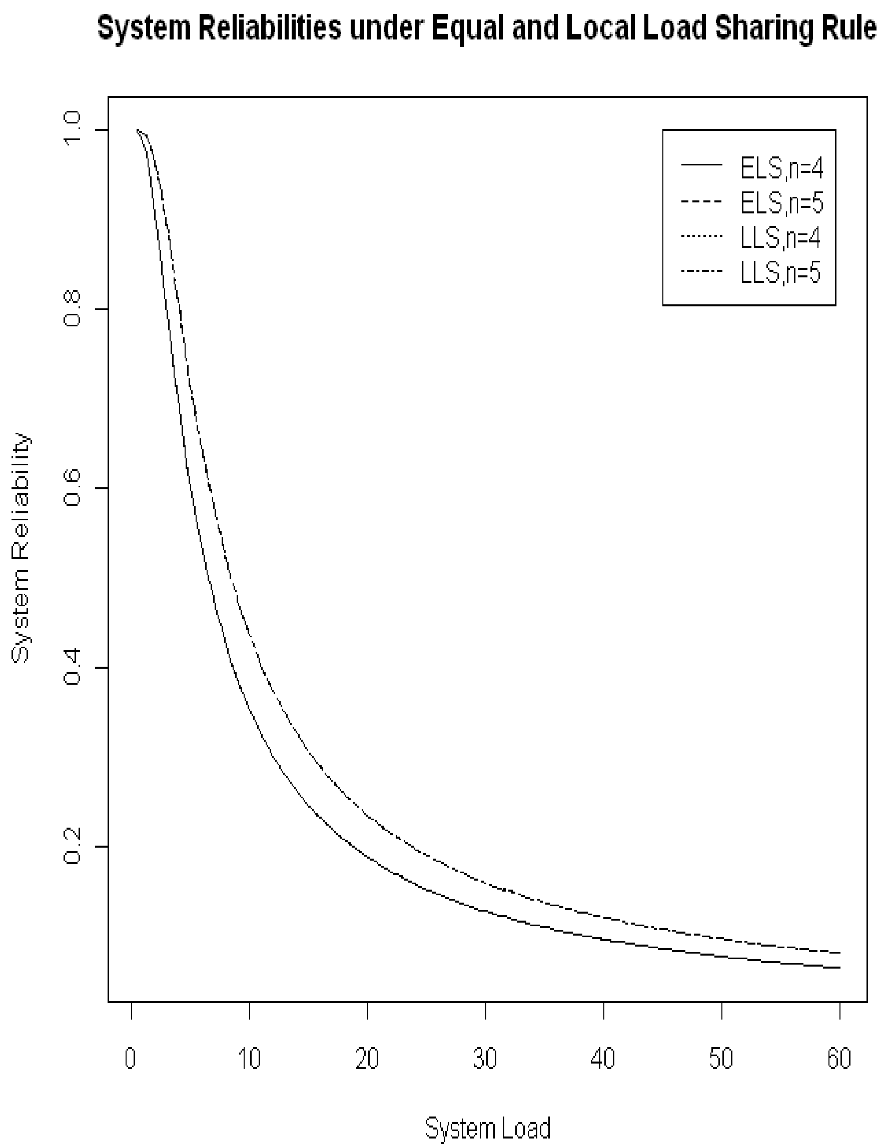

Figure 5.5: Static system reliability under ELS and LLS, for log-logistic component strength distributions with $\alpha = \beta = 1$.

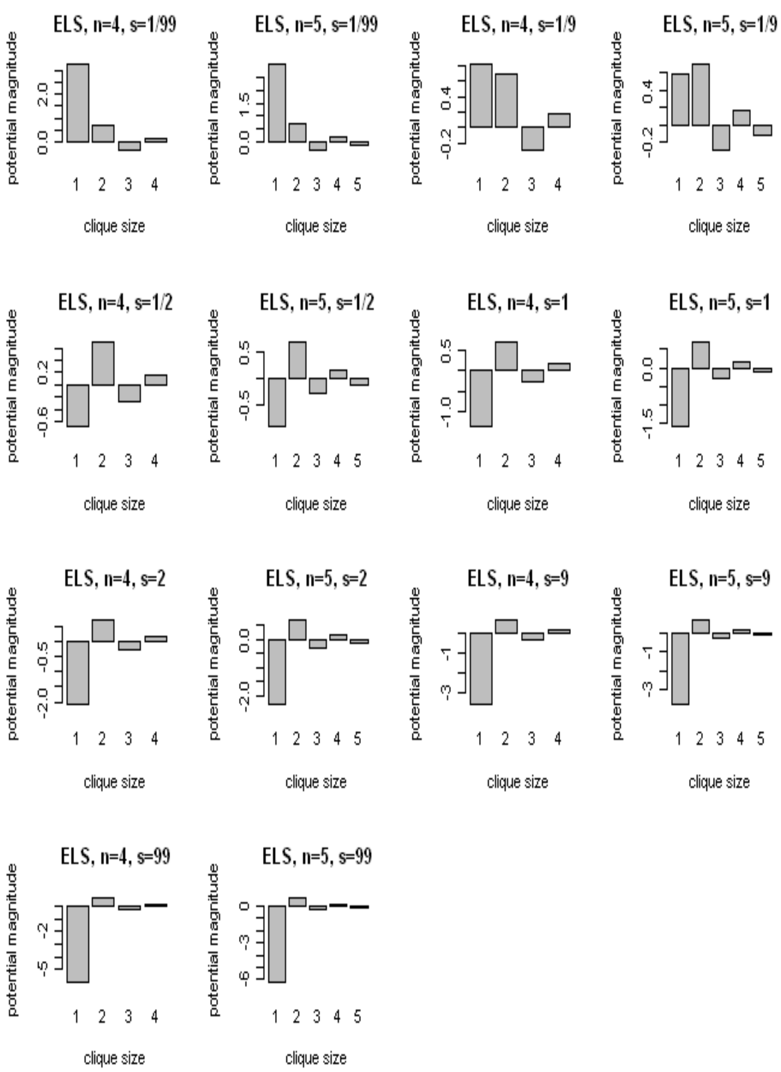

Figure 5.6: potentials under equal load-sharing with various s, $\alpha = \beta = 1$

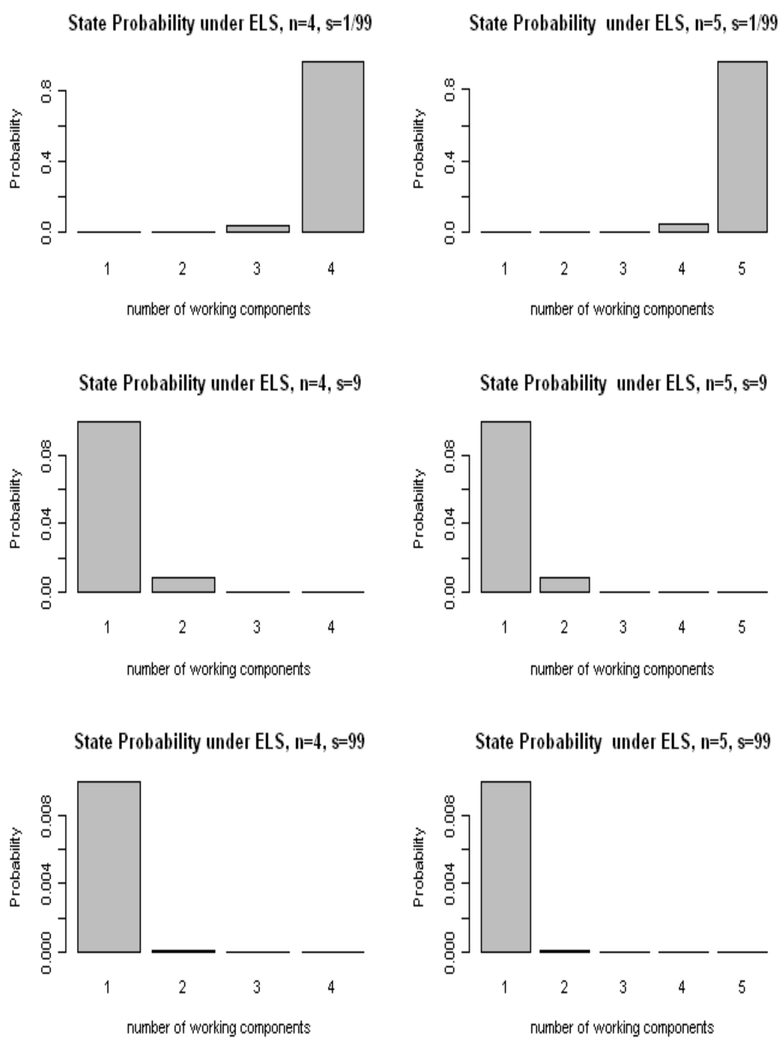

Figure 5.7: State probability under equal load-sharing with various s, $\alpha = \beta = 1$

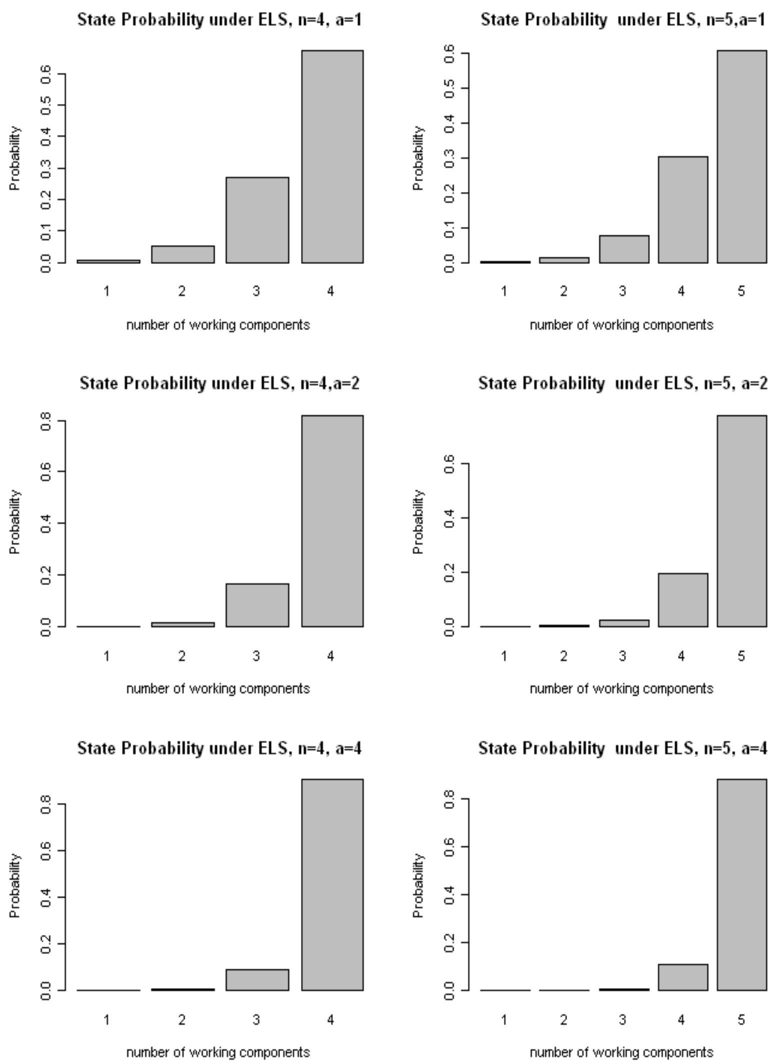

Figure 5.8: State probability, fix $s = 0.1$, $\beta = 1$, α varies $(\alpha = 1, 2, 4)$

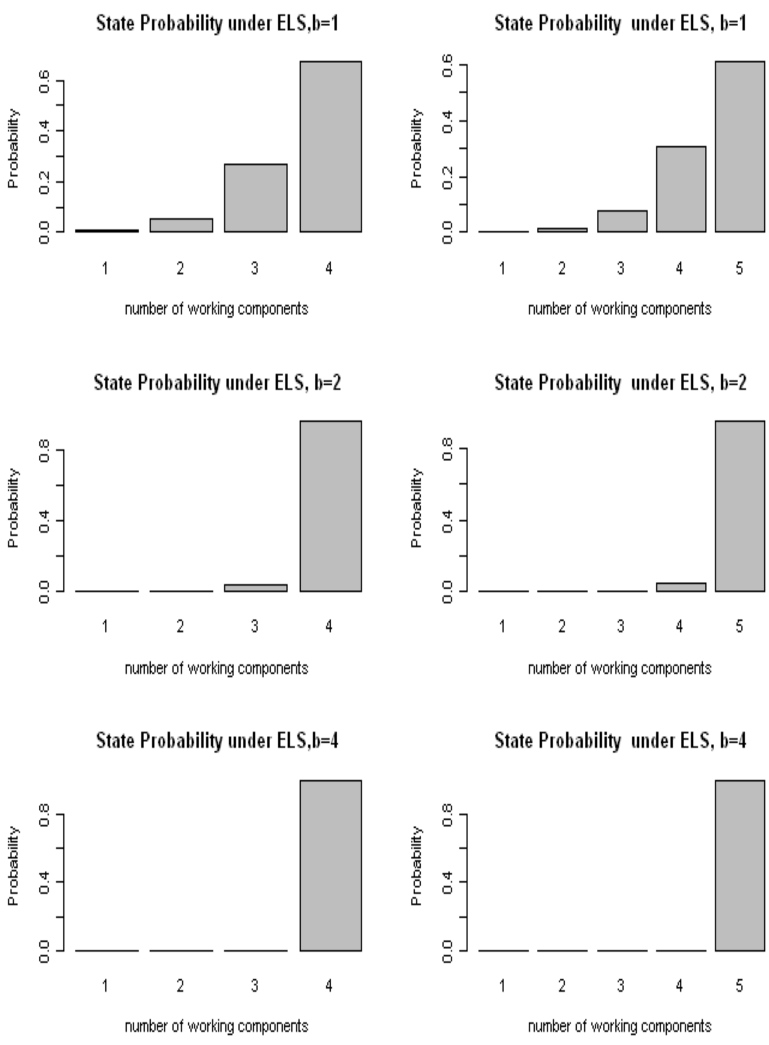

Figure 5.9: State probability, fix $s = 0.1$, $\alpha = 1$, β varies ($\beta = 1, 2, 4$)

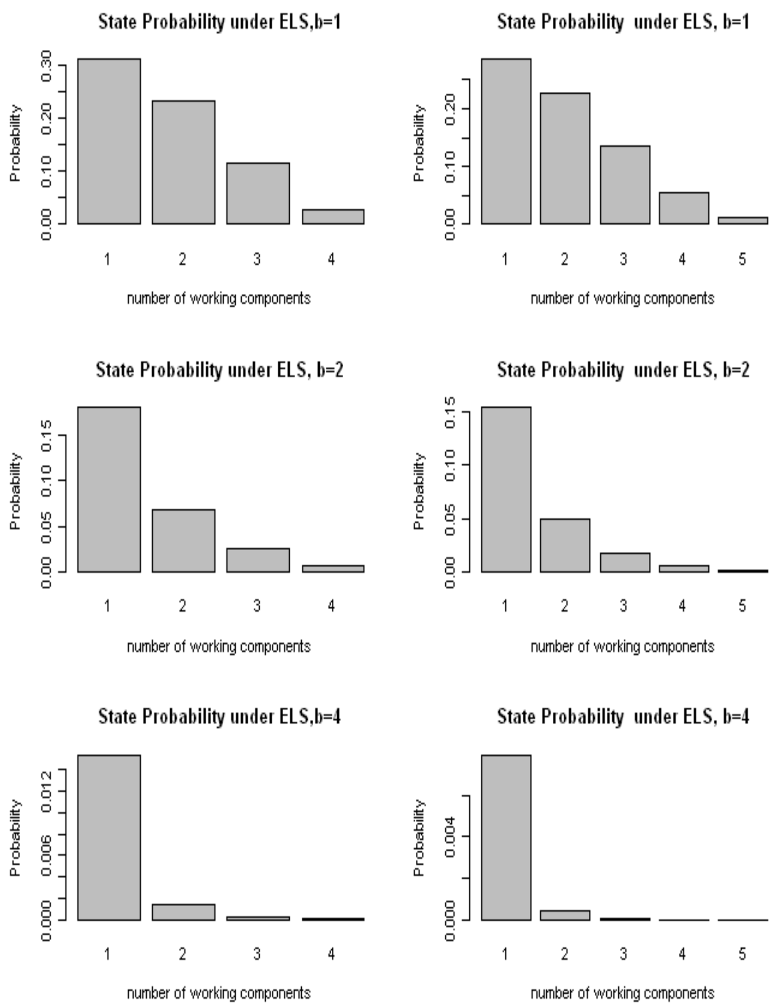

Figure 5.10: State probability, fix $s = 1, \alpha = 1$. β varies ($\beta = 1, 2, 4$)

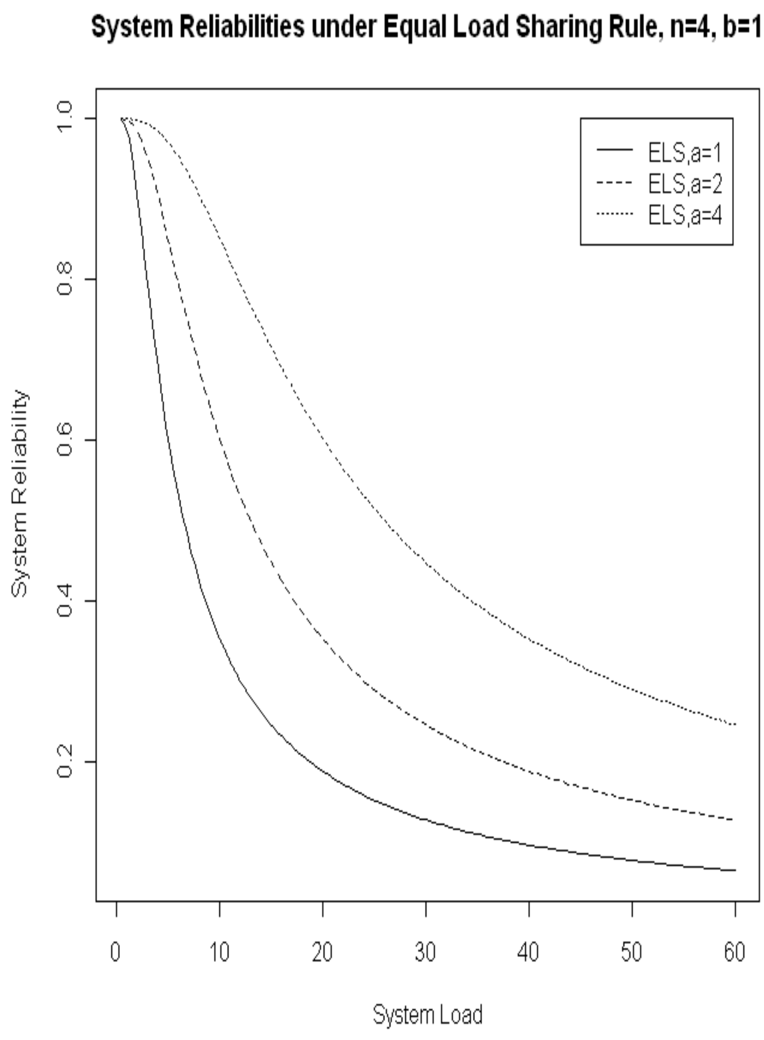

Figure 5.11: System reliability, fix $\beta = 1$, α varies ($\alpha = 1, 2, 4$)

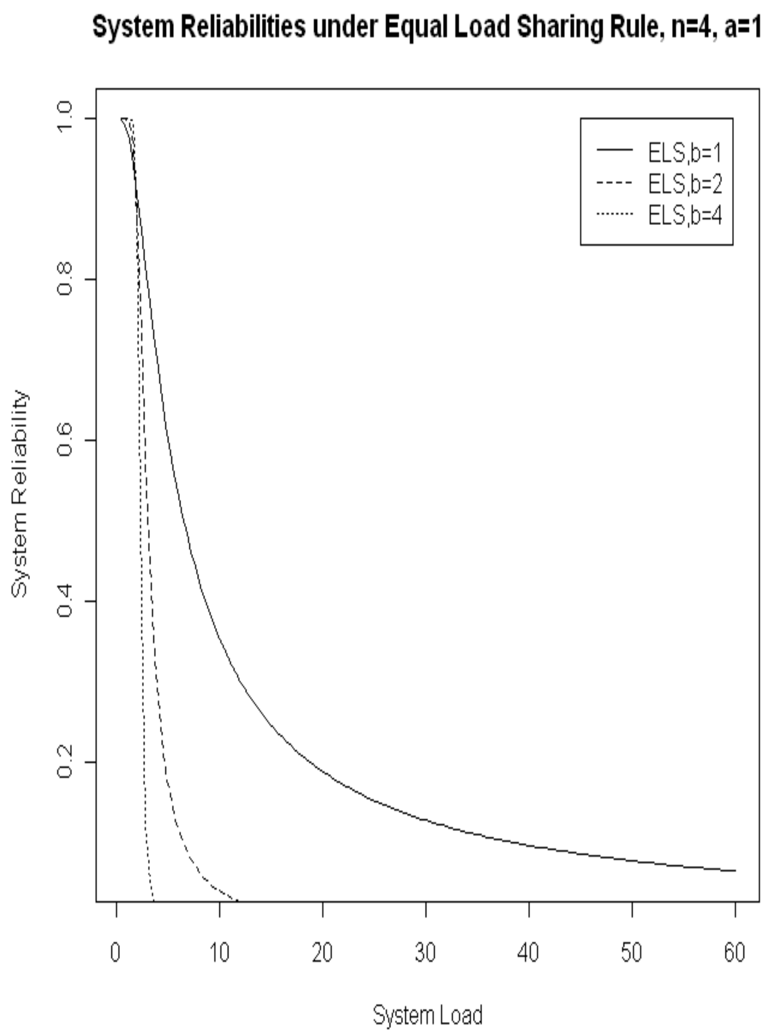

Figure 5.12: System reliability, fix $\alpha = 1$, β varies ($\beta = 1, 2, 4$)

6. Closing Comments

In this paper we have discussed some of the implications of representing the system state probabilities as a Gibbs measure. In particular, the potentials defining the Gibbs measure for a load-sharing parallel system are defined directly in terms of log odds of the component strength distributions and the load-sharing rule. The potentials indicate the intensity of the dependencies among components as a function of the stress s on the system.

To generalize the representations to non-parallel system is somewhat problematic because of the positivity condition (2.7). In particular, consider a set of working components A where A^c is a minimal cut set. Then the system has failed but components in A have not. For a physical system, such as Rosen's, the components in A are no longer under stress.

As an alternative, consider a minimal cut set C for the system. Since a minimal cut set is a parallel system, C has a Gibbs measure representation if infinitely strong components are placed in C^c. How satisfactory this formulation is the subject of further investigation. For mechanical systems, it is physically realizable by placing reinforced components in C^c and component testing regimes where cracks are created in specimens prior to failure testing.

Finally, let $\{\lambda_i(M) : M \subseteq N\}$ be a load-sharing rule where $\lambda_i(A) \neq \lambda_i(B)$ if $i \in A \subset B$ and let $G = (N, e)$ be an undirected graph. We can define a load-sharing system based on $\{\lambda_i(M) : i \in M \subseteq N\}$ that is compatible with G as follows.

Let N_i be the set of neighbors of component i given by G. For $A \subseteq N$ define $\tilde{\lambda}_i(A) = \lambda_i((A \cap N_i) \cup \{i\})$ and

$$P_s(Y_i = 1 \mid Y_j = 1,\ j \in A - \{i\},\ Y_k = 0,\ k \in A^c) = S_i(\tilde{\lambda}_i(A)s) \tag{6.1}$$

where we assume that the support of $S_i(x)$ is $[0, \infty)$ for all i. The following theorem shows that the load-sharing system on N with load-sharing rule $\{\tilde{\lambda}_i(M) : i \in M \subseteq N\}$ is compatible with a MRF on G.

Theorem 6.1

$$\begin{aligned} &P_s(Y_i = 1 \mid Y_j = 1,\ j \in A - \{i\},\ Y_k = 0,\ k \in A^c) \\ &\quad = P_s(Y_i = 1 \mid Y_j = 1,\ j \in A \cap N_i,\ Y_k = 0,\ k \in A^c \cap N_i) \end{aligned}$$

Proof: By (6.1)

$$\begin{aligned} &P_s(Y_i = 1 \mid Y_j = 1,\ j \in A - \{i\},\ Y_k = 0,\ k \in A^c) = S_i(\tilde{\lambda}_i(A)s) \\ &\quad = S_i(\lambda_i((A \cap N_i) \cup \{i\})s) \\ &\quad = S_i(\tilde{\lambda}_i((A \cap N_i) \cup \{i\})s) \\ &\quad = P_s(Y_i = 1 \mid Y_j = 1,\ j \in A \cap N_i,\ Y_k = 0,\ k \in (A \cap N_i)^c) \end{aligned}$$

Let B be a set such that $(A \cap N_i) \cup \{i\} \subseteq B \subseteq A$ (and so, $A^c \subseteq B^c \subseteq (A \cap N - i)^c$). Then $\tilde{\lambda}_i(B) = \tilde{\lambda}_i(A)$ and

$$\begin{aligned} & P_s(Y_i = 1 \mid Y_j = 1,\ j \in A - \{i\},\ Y_k = 0,\ k \in A^c) = S_i(\tilde{\lambda}_i(A)s) \\ = \;& S_i(\tilde{\lambda}_i(B)s) \\ = \;& P_s(Y_i = 1 \mid Y_j = 1,\ j \in B - \{i\},\ Y_k = 0,\ k \in B^c) \\ = \;& S_i(\tilde{\lambda}_i((A \cap N_i) \cup \{i\})s) \\ = \;& P_s(Y_i = 1 \mid Y_j = 1,\ j \in A \cap N_i,\ Y_k = 0,\ k \in (A \cap N_i)^c) \end{aligned}$$

Thus, from this sequence of identities it follows that the above conditional probabilities only depend on $Y_m, m \in (A \cap N_i) \cup A^c$.

Now suppose C is a set such that $C \subseteq A^c$ and $C \subseteq N_i^c$. Then

$$\begin{aligned}\tilde{\lambda}_i(A \cup C) &= \lambda_i(((A \cup C) \cap N_i) \cup \{i\}) \\ &= \lambda_i((A \cap N_i) \cup \{i\}) \\ &= \tilde{\lambda}_i(A)\end{aligned}$$

Thus,

$$\begin{aligned}& P_s(Y_i = 1 \mid Y_j = 1,\ j \in A \cup C - \{i\},\ Y_k = 0,\ k \in (A \cup C)^c) \\ =\ & S_i(\tilde{\lambda}_i(A \cup C)s) \\ =\ & S_i(\tilde{\lambda}_i(A)s) \\ =\ & P_s(Y_i = 1 \mid Y_j = 1,\ j \in A - \{i\},\ Y_k = 0,\ k \in A^c) \\ =\ & S_i(\tilde{\lambda}_i((A \cap N_i) \cup \{i\})s) \\ =\ & P_s(Y_i = 1 \mid Y_j = 1,\ j \in A \cap N_i,\ Y_k = 0,\ k \in (A \cap N_i)^c)\end{aligned}$$

From this sequence of identities it follows that the above conditional probabilities only depend on $Y_m, m \in A \cup (A^c \cap N_i)$.

Therefore, $P_s(Y_i = 1 \mid Y_j = 1,\ j \in A - \{i\},\ Y_k = 0,\ k \in A^c)$ depends only on Y_m with

$$m \in (A \cup (A^c \cap N_i)) \cap ((A \cap N - i) \cup A^c)) = N_i$$

which finishes the proof of Theorem 6.1. Hence the load-sharing rule $\{\tilde{\lambda}_i(M) : i \in M \subseteq N\}$ is compatible with a MRF that is described by the graph G. $\square$

Remark: By removing small potentials (thresholding the potentials) in the Gibbs measure representation one may get a less complex MRF than the trivial one at a given stress s for a load sharing system. In addition, as s is varied from the system median strength, one expects the MRF to become less complex. This may not be reflected in the approximate MRF's graph G but it is described by the potentials. It can be discerned in the graph G only if the resulting cliques in the approximate MRF give rise to a simpler neighborhood structure.

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