Product Kernels

Current posterior: $\pi(\theta)$

Current weighted-particle approximation: $\widehat{\pi}(\theta) = \sum_{i=1}^{N} w_i \delta_{\theta^i}$

As in LW, seek a continuous mixture approximation to $\widehat{\pi}(\theta)$
Goal: avoid parameter transformations

Example 1: Gamma kernels for variance parameters

θ positive parameter, e.g., a variance
Instead of Normal kernels, use Gamma kernels:

$$\text{Gam}(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{\beta x} \qquad x > 0$$

Moments and parameters

$$\begin{cases} \mu = \text{E}(X) = \dfrac{\alpha}{\beta} \\ \\ \sigma^2 = \text{Var}(X) = \dfrac{\alpha}{\beta^2} \end{cases} \qquad\qquad \begin{cases} \alpha = \dfrac{\mu^2}{\sigma^2} \\ \\ \beta = \dfrac{\mu}{\sigma^2} \end{cases}$$

Let $a \in (0, 1)$ be a smoothing parameter,

$$\bar{\theta} = \mathsf{E}_{\widehat{\pi}}(\theta) = \sum_i w_i \theta^i$$

$$S_\theta^2 = \mathsf{Var}_{\widehat{\pi}}(\theta) = \sum_i w_i (\theta^i - \bar{\theta})^2$$

For $i = 1, \ldots, N$, set

$$\mu_i = a\theta^i + (1 - a)\bar{\theta} \qquad \sigma_i^2 = (1 - a^2) S_\theta^2$$

and

$$\alpha_i = \frac{\mu_i^2}{\sigma_i^2} \qquad \beta_i = \frac{\mu_i}{\sigma_i^2}$$

Consider the mixture

$$\tilde{\pi}(\theta) = \sum_{i=1}^{N} w_i \, \mathsf{Gam}(\theta; \alpha_i, \beta_i)$$

Then

1. $\tilde{\pi}(\theta)$ is continuous

2. $\mathsf{Supp}(\tilde{\pi}) = (0, +\infty)$.

3. $\mathsf{E}_{\hat{\pi}}(\theta) = \mathsf{E}_{\tilde{\pi}}(\theta)$ and $\mathsf{Var}_{\hat{\pi}}(\theta) = \mathsf{Var}_{\tilde{\pi}}(\theta)$

Example 2: Products of Gamma kernels

$\theta = (\theta_1, \theta_2)$ two positive parameters, e.g., two variances
Weighted-particle approximation

$$\widehat{\pi}(\theta) = \sum_{i=1}^{N} w_i \delta_{\theta^i}, \qquad \theta^i = (\theta_1^i, \theta_2^i)$$

Let $a \in (0, 1)$ be a smoothing parameter,

$$\bar{\theta} = \mathsf{E}_{\widehat{\pi}}(\theta) = (\bar{\theta}_1, \bar{\theta}_2)$$

$$S^2 = \mathsf{Var}_{\widehat{\pi}}(\theta) = \sum_i w_i (\theta^i - \bar{\theta})'(\theta^i - \bar{\theta}) = \begin{bmatrix} S_1^2 & S_{12}^2 \\ S_{21}^2 & S_2^2 \end{bmatrix}$$

For $k = 1, 2$ and $i = 1, \ldots, N$, set

$$\mu_{ki} = a\theta_k^i + (1 - a)\bar{\theta}_k \qquad \sigma_{ki}^2 = (1 - a^2)S_k^2$$

and

$$\alpha_{ki} = \frac{\mu_{ki}^2}{\sigma_{ki}^2} \qquad \beta_{ki} = \frac{\mu_{ki}}{\sigma_{ki}^2}$$

That is, proceed as in Example 1, but marginally for θ_1 and θ_2

Consider the mixture

$$\tilde{\pi}(\theta) = \tilde{\pi}(\theta_1, \theta_2) = \sum_{i=1}^{N} w_i \, \mathrm{Gam}(\theta_1; \alpha_{1i}, \beta_{1i}) \, \mathrm{Gam}(\theta_2; \alpha_{2i}, \beta_{2i})$$

Then

1. $\tilde{\pi}(\theta)$ is continuous

2. $\mathrm{Supp}(\tilde{\pi}) = (0, +\infty) \times (0, +\infty)$.

3. $\mathrm{E}_{\tilde{\pi}}(\theta) = \mathrm{E}_{\hat{\pi}}(\theta)$ and $\mathrm{Var}_{\tilde{\pi}}(\theta) = \begin{bmatrix} S_1^2 & a^2 S_{12}^2 \\ a^2 S_{21}^2 & S_2^2 \end{bmatrix}$

Note that

$$\mathsf{diag}(\mathsf{Var}_{\tilde{\pi}}(\theta)) = \mathsf{diag}(\mathsf{Var}_{\hat{\pi}}(\theta))$$

and

$$\frac{\mathsf{Cov}_{\tilde{\pi}}(\theta_1, \theta_2)}{\mathsf{Cov}_{\hat{\pi}}(\theta_1, \theta_2)} = a^2 \quad \left(\approx 1,\ \text{typically}\right)$$

Other potentially useful building-block kernels include

- Beta, for probabilities or parameters supported on a finite interval
- Wishart, for unknown static variance matrices
- ...?