APF, SIR & LW

- Consider the following state space model

$$\begin{aligned} \text{Observation equation} &: \quad p(y_{t+1}|x_{t+1},\theta) \\ \text{State equation} &: \quad p(x_{t+1}|x_t,\theta) \\ \text{Fixed parameters} &: \quad p(\theta) \end{aligned}$$

- For a given time t

$$\{(x_t,\theta,\omega_t)^{(i)}\}_{i=1}^N$$

is a particle representation of $p(x_t,\theta|y^t)$.

- $y^t = (y_1,\ldots,y_n)$

APF with known θ

- Pitt and Shephard (1999) propose augmenting

$$p^N(x_{t+1}, k|y^{t+1}) \propto p(x_{t+1}|x_t^{(k)})p(y_{t+1}|x_{t+1})\omega_t^{(k)}$$

- Proposal

$$q(x_{t+1}, k|y^{t+1}) \propto p(x_{t+1}|x_t^{(k)})p(y_{t+1}|g(x_t^{(k)}))\omega_t^{(k)}$$

from which $\{(\tilde{k}, \tilde{x}_{t+1})\}_{i=1}^N$ are generated. Also,

$$g(x_t) = E(x_{t+1}|x_t)$$

- New weights

$$\tilde{\omega}_{t+1} \propto \frac{p(y_{t+1}|\tilde{x}_{t+1})}{p(y_{t+1}|g(x_t^{\tilde{k}}))}$$

SIR argument with known θ

The objective is to obtain

$$\{(x_t, x_{t+1}, \omega_{t+1})^{(i)}\}_{i=1}^N \sim p(x_t, x_{t+1}|y^{t+1})$$

based on $\{(x_t, \omega_t)^{(i)}\}_{i=1}^N \sim p(x_t|y^t)$.

- Resample $x_t^* \sim q_1(x_t|y_{t+1})$
- Propagate $\tilde{x}_{t+1} \sim q_2(x_{t+1}|x_t^*, y_{t+1})$
- Weight

$$\tilde{\omega}_{t+1} \propto \frac{\omega_t p(y_{t+1}|\tilde{x}_{t+1})}{q_1(x_t^*|y_{t+1})} \frac{p(\tilde{x}_{t+1}|x_t^*)}{q_2(\tilde{x}_{t+1}|x_t^*, y_{t+1})}$$

- Notes:
  - How to choose q_1 and q_2?
  - APF: $q_1 = \omega_t p(y_{t+1}|g(x_t^*))$ and $q_2 = p(\tilde{x}_{t+1}|x_t^*)$.

Perfect adaptation

- Weights are constant when

$$\begin{aligned} q_1(x_t|y_{t+1}) &= p(x_t|y_{t+1}) \\ q_2(x_{t+1}|x_t, y_{t+1}) &= p(x_{t+1}|x_t, y_{t+1}). \end{aligned}$$

- Both q_1 and q_2 depend on y_{t+1}.

SIR with learning of θ

The objective is to obtain

$$\{(x_t, x_{t+1}, \theta, \omega_{t+1})^{(i)}\}_{i=1}^N \sim p(x_t, x_{t+1}, \theta | y^{t+1})$$

based on $\{(x_t, \theta, \omega_t)^{(i)}\}_{i=1}^N \sim p(x_t, \theta | y^t)$.

- Resample $(x_t^*, \theta^*) \sim q_1(x_t, \theta | y_{t+1})$
- Propagate $(\tilde{x}_{t+1}, \tilde{\theta}) \sim q_2(x_{t+1}, \theta | x_t^*, \theta^*, y_{t+1})$
- Weight

$$\tilde{\omega}_{t+1} \propto \frac{\omega_t p(y_{t+1} | \tilde{x}_{t+1}, \tilde{\theta})}{q_1(x_t^*, \theta^* | y_{t+1})} \frac{p(\tilde{x}_{t+1}, \tilde{\theta} | x_t^*, \theta^*)}{q_2(\tilde{x}_{t+1}, \tilde{\theta} | x_t^*, \theta^*, y_{t+1})}$$

- Notes:
  - How to choose q_1 and q_2?
  - Is $p(x_{t+1}, \theta | x_t^*, \theta^*) = p(x_{t+1} | \theta, x_t^*) p(\theta | x_t^*, \theta^*)$?
  - If so, then what is $p(\theta | x_t^*, \theta^*)$?

Liu and West (2001)

They chose $p(\theta|x_t^*,\theta^*)$, q_1 and q_2 to be

$$
\begin{aligned}
p(\theta|x_t^*,\theta^*) &= f_N(\theta|m(\theta^*),h^2V) \\
q_1(x_t^*,\theta^*|y_{t+1}) &= \omega_t p(y_{t+1}|g(x_t^*),m(\theta^*)) \\
q_2(x_{t+1},\theta|x_t^*,\theta^*,y_{t+1}) &= q_{21}(\theta|x_t^*,\theta^*,y_{t+1})q_{22}(x_{t+1}|\theta,x_t^*,\theta^*,y_{t+1}) \\
&= f_N(\theta|m(\theta^*),h^2V)p(x_{t+1}|\theta,x_t^*) \\
m(\theta^*) &= a\theta^* + (1-a)\bar{\theta} \\
\bar{\theta} &= \text{particle mean of } \theta \\
V &= \text{particle variance of } \theta \\
a^2 &= 1-h^2 \\
h^2 &= 1-((3\delta-1)/(2\delta))^2
\end{aligned}
$$

so weights are

$$\tilde{\omega}_{t+1} \propto \frac{p(y_{t+1}|\tilde{x}_{t+1},\tilde{\theta})}{p(y_{t+1}|g(x_t^*),m(\theta^*))}$$

- **Note:** Neither q_{21} or q_{22} depend on y_{t+1}.

Simulation exercise

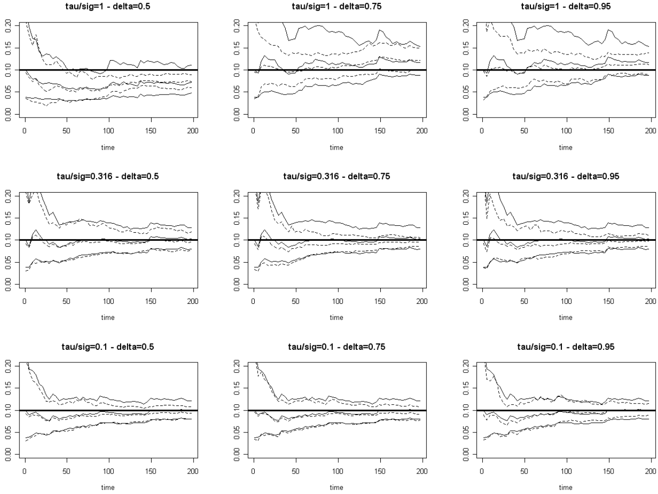
- Simulation set up

$$
\begin{aligned}
y_t &= x_t + \nu_t \qquad \nu_t \sim N(0, 0.1) \\
x_t &= x_{t-1} + \omega_t \qquad \omega_t \sim N(0, \tau^2) \\
x_0 &= 25 \\
\tau^2 &= (0.1, 0.01, 0.001) \\
\delta &= (0.5, 0.75, 0.95) \\
\text{Sample sizes} &= (200, 500) \\
\text{Particles} &= (2000, 5000)
\end{aligned}
$$

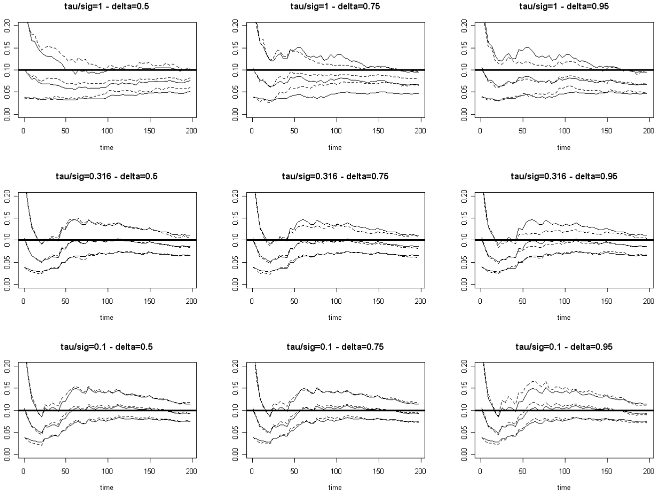
- Model set up

$$
\begin{aligned}
y_t &= x_t + \nu_t \qquad \nu_t \sim N(0, \sigma^2) \\
x_t &= x_{t-1} + \omega_t \qquad \omega_t \sim N(0, \tau^2) \\
\sigma^2 &\sim IG(5, 0.4) \\
x_0 &\sim N(25, 100)
\end{aligned}
$$

T=200, M=2000

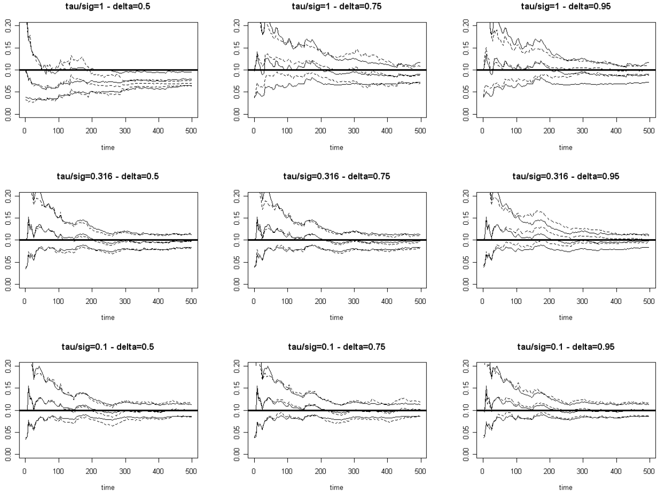

tau/sig=1 - delta=0.5
tau/sig=1 - delta=0.75
tau/sig=1 - delta=0.95
tau/sig=0.316 - delta=0.5
tau/sig=0.316 - delta=0.75
tau/sig=0.316 - delta=0.95
tau/sig=0.1 - delta=0.5
tau/sig=0.1 - delta=0.75
tau/sig=0.1 - delta=0.95
0.00
0.05
0.10
0.15
0.20
0
50
100
150
200
time

T=200, M=5000



tau/sig=1 - delta=0.5
tau/sig=1 - delta=0.75
tau/sig=1 - delta=0.95
tau/sig=0.316 - delta=0.5
tau/sig=0.316 - delta=0.75
tau/sig=0.316 - delta=0.95
tau/sig=0.1 - delta=0.5
tau/sig=0.1 - delta=0.75
tau/sig=0.1 - delta=0.95
time

T=500, M=5000


tau/sig=1 - delta=0.5
tau/sig=1 - delta=0.75
tau/sig=1 - delta=0.95
tau/sig=0.316 - delta=0.5
tau/sig=0.316 - delta=0.75
tau/sig=0.316 - delta=0.95
tau/sig=0.1 - delta=0.5
tau/sig=0.1 - delta=0.75
tau/sig=0.1 - delta=0.95
0.00
0.05
0.10
0.15
0.20
0
100
200
300
400
500
time

Thoughts

- $\delta = 0.75$ looks robust across T, M and τ/σ;

- The difference between PL and APF decreases with τ/σ.
  Conjecture: Probably partially because of the choice

$$q_{22}(x_{t+1}|\theta, x_t^*, \theta^*, y_{t+1}) = p(x_{t+1}|\theta^*, x_t^*)$$

  which is the usual *blind* propagation rule used in standard SIS-based filters.

- If so, then maybe

$$q_{21}(\theta|x_t^*, \theta^*, y_{t+1}) = f_N(\theta|m(\theta^*), h^2 V)$$

  is not such a bad choice!

- This will be my simulation exercise for next week.

Modifying q_{21} and q_{22}

- LW1

$$q_{22}(x_{t+1}|x_t, \theta, y_{t+1}) = f_N(x_{t+1}|m_{t+1}, C_{t+1})$$

where

$$\begin{aligned} C_{t+1}^{-1} &= \tau^{-2} + \sigma^{-2} \\ m_{t+1} &= C_{t+1}(\sigma^{-2} y_{t+1} + \tau^{-2} x_t) \end{aligned}$$

- LW2 = LW1 and

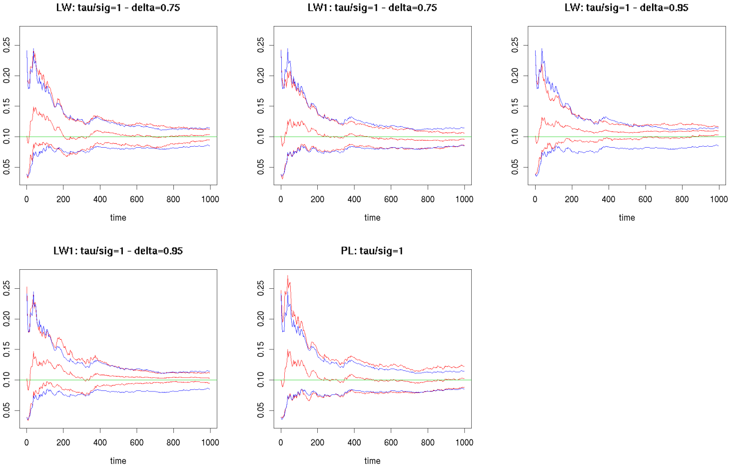
$$q_{21}(\sigma^2|x_t^*, \sigma^{2*}, y_{t+1}) \propto f_N(y_{t+1}; x_t^*, \sigma^2) f_{IG}(\sigma^2|\alpha(\sigma^{2*}), \beta(\sigma^{2*}))$$

where

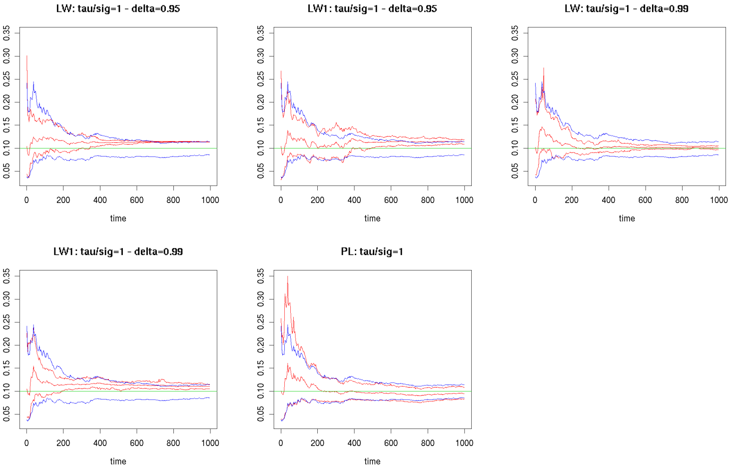
$$\begin{aligned} \alpha(\sigma^{2*}) &= m(\sigma^{2*})^2/v(\sigma^{2*}) + 2 \\ \beta(\sigma^{2*}) &= m(\sigma^{2*})\alpha(\sigma^{2*}) \\ m(\sigma^{2*}) &= a\sigma^{2*} + (1-a)\bar{\sigma}^2 \\ v(\sigma^{2*}) &= h^2 S_{\sigma^2}^2 \end{aligned}$$

with $\bar{\sigma}^2$ and $S_{\sigma^2}^2$ mean and variance of the particles for σ^2.

M=10000



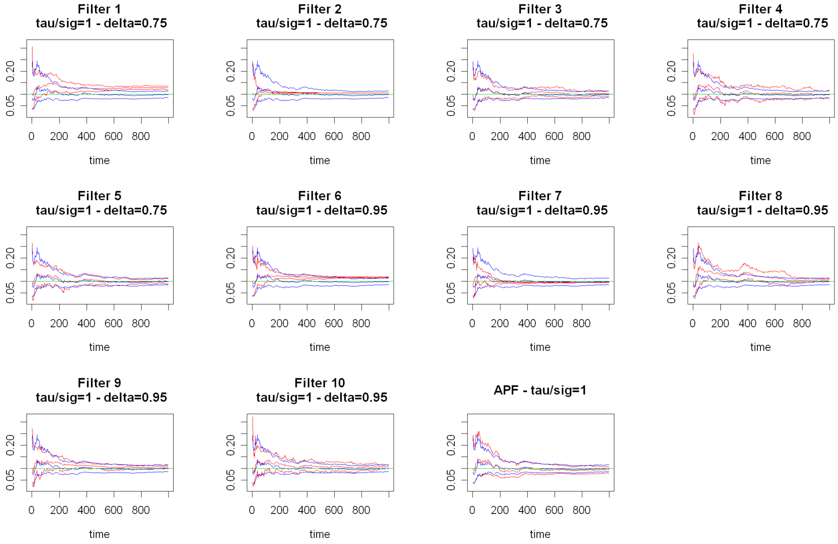
LW: tau/sig=1 - delta=0.75
LW1: tau/sig=1 - delta=0.75
LW: tau/sig=1 - delta=0.95
LW1: tau/sig=1 - delta=0.95
PL: tau/sig=1
0.05
0.10
0.15
0.20
0.25
0
200
400
600
800
1000
time

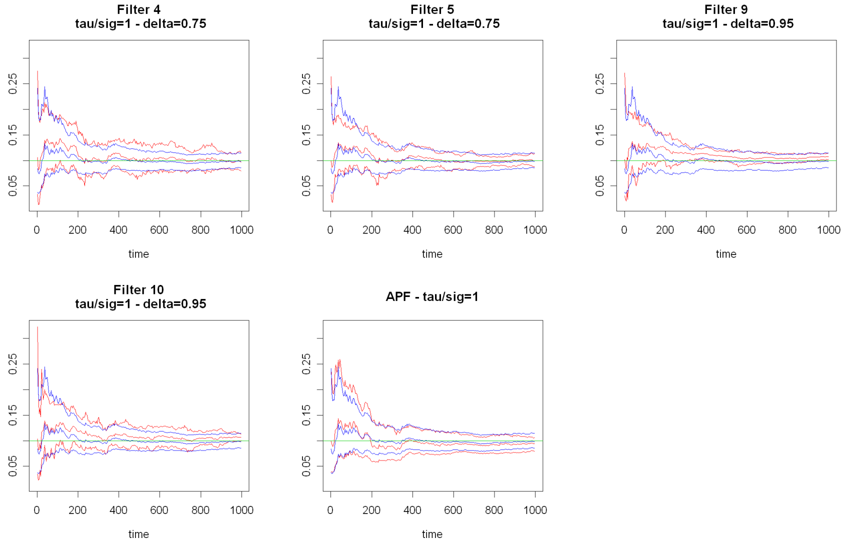

LW: tau/sig=1 - delta=0.95
LW1: tau/sig=1 - delta=0.95
LW: tau/sig=1 - delta=0.99
LW1: tau/sig=1 - delta=0.99
PL: tau/sig=1
time
0.05 0.10 0.15 0.20 0.25 0.30 0.35
0 200 400 600 800 1000

Various filters

- Filter 1: LW: $\log \sigma^2$ and $\delta = 0.75$
- Filter 2: LW: σ^2 and $\delta = 0.75$
- Filter 3: LW1: $\log \sigma^2$ and $\delta = 0.75$
- Filter 4: LW1: σ^2 and $\delta = 0.75$
- Filter 5: LW2: σ^2 and $\delta = 0.75$
- Filter 6: LW: $\log \sigma^2$ and $\delta = 0.95$
- Filter 7: LW: σ^2 and $\delta = 0.95$
- Filter 8: LW1: $\log \sigma^2$ and $\delta = 0.95$
- Filter 9: LW1: σ^2 and $\delta = 0.95$
- Filter 10: LW2: σ^2 and $\delta = 0.95$
- Filter 11: PL

M=2000


Filter 1
tau/sig=1 - delta=0.75
Filter 2
tau/sig=1 - delta=0.75
Filter 3
tau/sig=1 - delta=0.75
Filter 4
tau/sig=1 - delta=0.75
Filter 5
tau/sig=1 - delta=0.75
Filter 6
tau/sig=1 - delta=0.95
Filter 7
tau/sig=1 - delta=0.95
Filter 8
tau/sig=1 - delta=0.95
Filter 9
tau/sig=1 - delta=0.95
Filter 10
tau/sig=1 - delta=0.95
APF - tau/sig=1
0.05
0.20
0
200
400
600
800
time


Filter 4
tau/sig=1 - delta=0.75
Filter 5
tau/sig=1 - delta=0.75
Filter 9
tau/sig=1 - delta=0.95
Filter 10
tau/sig=1 - delta=0.95
APF - tau/sig=1
0.05
0.15
0.25
0
200
400
600
800
1000
time

References

- Godsill, S. and Clapp, T. (2001). Improvement strategies for Monte Carlo particle filters. In *Sequential Monte Carlo Methods in Practice*. Springer-Verlag New York.

- Liu, J. and West, M. (2001). Combined parameters and state estimation in simulation-based filtering. In *Sequential Monte Carlo Methods in Practice*. Springer-Verlag New York.

- Johansen, A. and Doucet, A. (2008). A note on auxiliary particle filters. *Statistics and Probability Letters* (to appear).

- Pitt, M. and Shephard, N. (1999) Filtering via simulation: Auxiliary particle filters. *Journal of the American Statistical Association*, 94, 590-599.