Zakharov–Shabat Eigenvalue Problem

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -i\lambda & q(t) \\ -q(t)^* & i\lambda \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \qquad \left(L(q)\begin{bmatrix} u \\ v \end{bmatrix} = \lambda\begin{bmatrix} u \\ v \end{bmatrix} \right)$$

real.

$q : \mathbb{R} \to \mathbb{C}$, rapidly decreasing to $q=0$ for large $|t|$.

$\lambda \in \mathbb{C}$ is the spectral parameter.

L is nonselfadjoint on $L^2(\mathbb{R})$
non-normal

Problem: find L^2 eigenvalues λ.

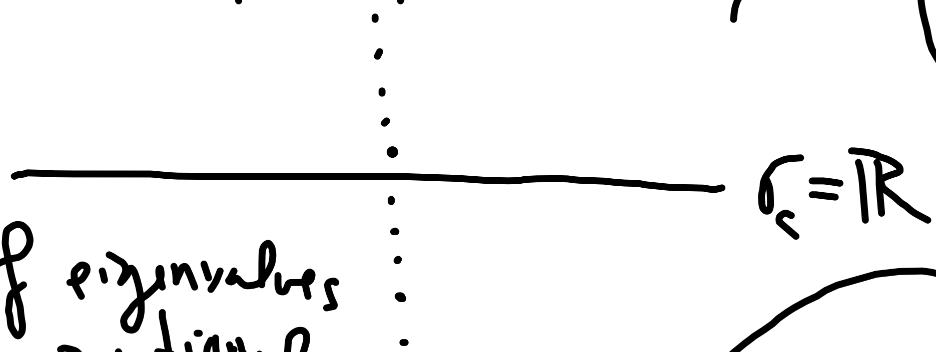
One result (Klaus & Shaw): if $q(t)$ is

① real-valued (more generally, constant phase)

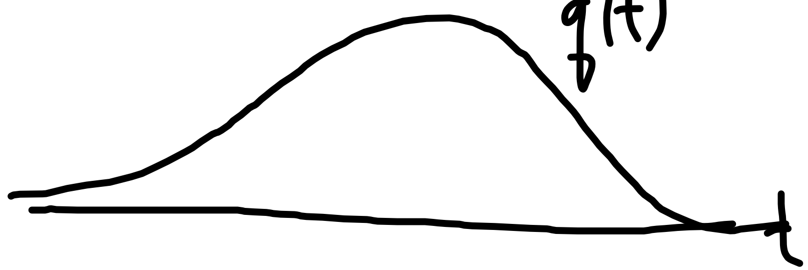
② "single-lobe" (only one critical pt, a local maximum)

then σ_p is purely imaginary.

$\lambda \in \sigma \Rightarrow -\bar{\lambda} \in \sigma$ for all q



Also, # of eigenvalues is proportional to $\dfrac{\|q\|_1}{\varepsilon}$



Restrict to Klaus-Shaw potentials $q(t)$.

$\varepsilon \to 0$ is like dimension of a matrix $\to \infty$.

"Random Matrix Limit": $\varepsilon \to 0$
$q(t)$ random K.S.

Consider first $q(t)$ non-random, $\varepsilon \to 0$.

We know $r_p \in i\mathbb{R}$, so set $s = -i\lambda \in \mathbb{R}_+$

$$\varepsilon \frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} s & q(t) \\ -q(t) & -s \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

We consider a special case:

Special case (Satsuma + Yajima)

$$\mu \frac{d}{dx}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} r & \operatorname{sech}(x) \\ -\operatorname{sech}(x) & -r \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}, \quad \mu \in \mathbb{R},\ r \in \mathbb{R}_+$$

Convert (rewrite) as a hypergeometric eqn:

$$y = \tanh(x)$$

$$\operatorname{sech}(x) = \sqrt{1-\tanh^2(x)} = \sqrt{1-y^2}$$

$$\frac{d}{dx} = \frac{dy}{dx}\frac{d}{dy} = \operatorname{sech}^2(x)\frac{d}{dy} = (1-y^2)\frac{d}{dy}$$

$$\mu(1-y^2)\frac{d}{dy}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} r & \sqrt{1-y^2} \\ -\sqrt{1-y^2} & -r \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

- Simple transformation removes the square roots $\Longrightarrow$ Linear 2×2 (second-order) system with rational coefficients.

- There are exactly 3 singular points all <u>regular</u>.

<u>Defn:</u> $\vec{u}' = \underline{\underline{M}}(x)\vec{u}$, entries of $\underline{\underline{M}}$ rational in x.

<u>Method of Frobenius</u>

x_0 is a pole of $\underline{\underline{M}}$: $\underline{\underline{M}} = (x-x_0)^{-1}\underline{\underline{M}}_0 + \ldots$ Laurent expansion.

Try: $u = (x-x_0)^{\rho}\sum_{n=0}^{\infty}(x-x_0)^n \vec{u}_n$

Leading term: RHS: $(x-x_0)^{\rho-\ell}\,\underline{\underline{M}}_0\cdot\vec{u}_0$

LHS: $\rho(x-x_0)^{\rho-1}\,\vec{u}_0$

$\underline{\ell=1}$

$$\underline{\underline{M}}_0\vec{u}_0=\rho\vec{u}_0$$

$\underline{\ell=2}$

$$\underline{\underline{M}}_0\vec{u}_0=\vec{0} \qquad (\text{if } \underline{\underline{M}}_0 \text{ is singular})$$

$\underline{\ell\geq 3}$ Frobenius fails to produce even one nontrivial solution.

<u>Irregular singular</u>

Upshot: $\mu \frac{d}{dx}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} r & \operatorname{sech}(x) \\ -\operatorname{sech}(x) & -r \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$

can be solved (i.e. a basis of solutions can be found [in the form of integrals]) for all $\mu \in \mathbb{R}$, $r \in \mathbb{C}$.

$$\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-y^2} \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \vec{b}(y)$$

$$\mu(1-y^2)\frac{d\vec{b}}{dy} = \begin{bmatrix} r & 1 \\ -(1-y^2) & -(r+\mu y) \end{bmatrix}\vec{b}$$

$$\vec{f}(y) := \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{1-y^2} \end{bmatrix} (1+y)^{-r/2\mu} (1-y)^{r/2\mu} \vec{b}(y)$$

$$\| $$

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$\mu \frac{df_1}{dy} = f_2 \, , \quad \mu \frac{df_2}{dy} = \frac{-f_1 + (\mu y - 2r) f_2}{1-y^2}$$

Solve with Euler transforms

$$\alpha - 1 = \beta$$

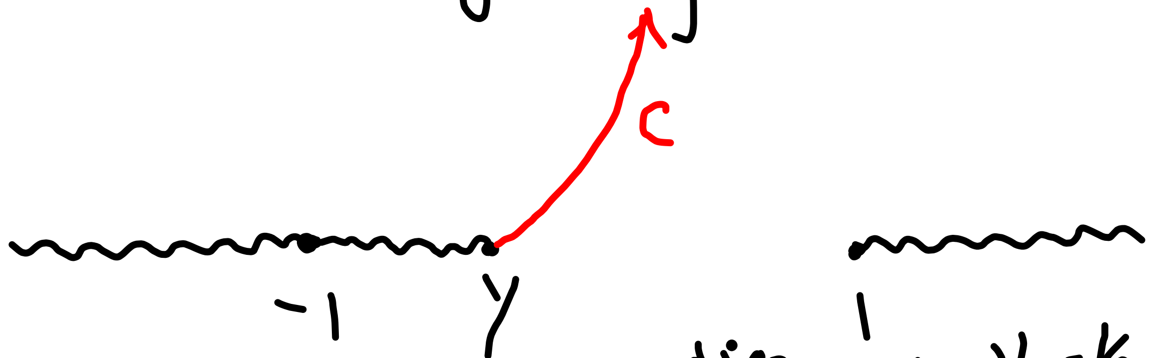
$$f_1(y) = \int_C F_1(t)(t-y)^{\alpha} dt \qquad -\mu\alpha F_1(t) = F_2(t)$$

$$f_2(y) = \int_C F_2(t)(t-y)^{\beta} dt$$

$\alpha = \frac{1}{\mu}$. Remove several terms.

$$\frac{d}{dt}\log F_1 = \frac{d}{dt}\log\left((1-t)^{-\alpha-\frac{1}{2}+r/\mu}\,(1+t)^{-\alpha-\frac{1}{2}-r/\mu}\right)$$

Paths of integration



$$f_k(y) = (-1)^{k+1}\int_y^{+i\infty}(1-t)^{-(1-r)/\mu-1/2}(1+t)^{-(1+r)/\mu-\frac{1}{2}}(t-y)^{1/\mu+1-k}\,dt$$

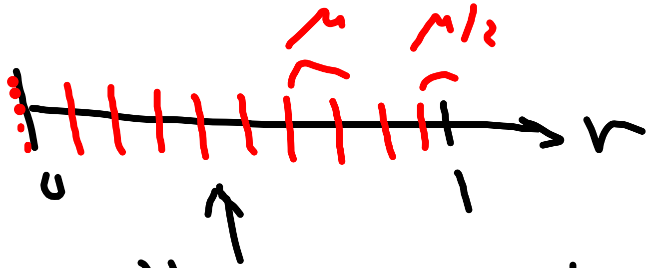
Get real-valued soln's by taking Re, Im.

Eigenvalue Condition

$$\frac{r}{\mu} - \frac{1}{\mu} - \frac{1}{2} \in \mathbb{Z}, \quad r > 0.$$

$$r_k = 1 - \left(k + \tfrac{1}{2}\right)\mu, \quad k = 0, 1, \ldots, N-1$$

$$N = \left\lceil \frac{1}{\mu} \right\rceil$$



"Deterministic level repulsion"

General potentials $q(t)$

Not hypergeometric, but ε is small.

$$\varepsilon \frac{d}{dt}\vec{u} = \begin{bmatrix} s & q(t) \\ -q(t) & -s \end{bmatrix} \vec{u}$$

$$\vec{v} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \vec{u} \qquad \text{unitary}$$

$$\varepsilon \frac{d\vec{v}}{dt} = \begin{bmatrix} 0 & s - q(t) \\ s + q(t) & 0 \end{bmatrix} \vec{v}$$

Goal: find a change of indep/dep variables $(t, \vec{v})$ (Langer transf.) so this problem becomes a perturbation of sech case.

$$t = t(x) \quad , \quad \vec{v} = \underline{\underline{A}}\,\vec{w}$$

$$\varepsilon \frac{dx}{dt} \frac{d}{dx}\left(\underline{\underline{A}}\,\vec{w}\right) = \begin{bmatrix} 0 & s - q(t) \\ s + q(t) & 0 \end{bmatrix} \underline{\underline{A}}\,\vec{w}$$

$$\varepsilon \frac{dA}{dx}\vec{w} + \varepsilon A \frac{d\vec{w}}{dx} = \frac{dt}{dx}\begin{bmatrix} 0 & s - q \\ s + q & 0 \end{bmatrix} A\vec{w}$$

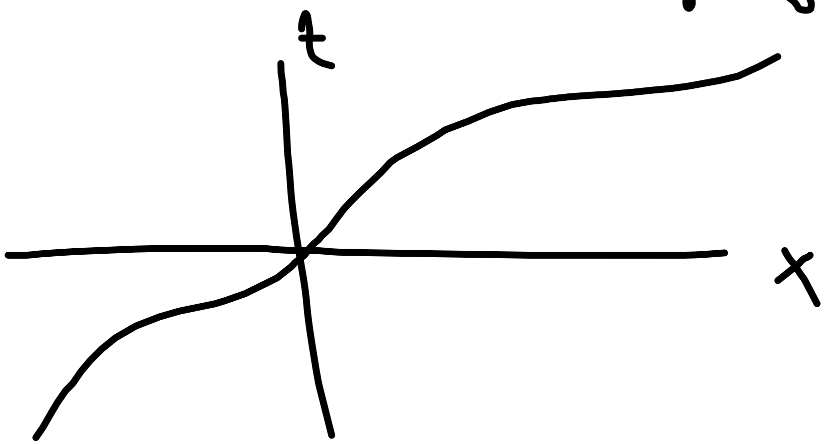
$$\varepsilon \frac{d\vec{w}}{dx} = \underbrace{A^{-1} \frac{dt}{dx}\begin{bmatrix} 0 & s - q \\ s + q & 0 \end{bmatrix}}_{\text{make this become}} A\vec{w} - \varepsilon A^{-1}\frac{dA}{dx}\vec{w}$$

make this become

$$c \cdot \begin{bmatrix} 0 & r - \operatorname{sech}(x) \\ r + \operatorname{sech}(x) & 0 \end{bmatrix} \Rightarrow \left(\frac{dt}{dx}\right)^2 (q^2 - s^2) = c^2 \cdot \left(\operatorname{sech}^2(x) - r^2\right)$$

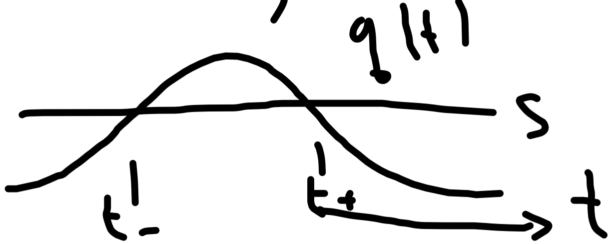
Need to solve $\left(\frac{dt}{dx}\right)^2 \left(q(t)^2 - s^2 \right) = c^2 \left(\operatorname{sech}^2(x) - r^2 \right)$

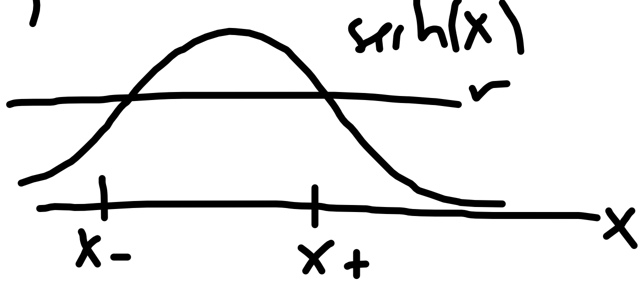
for a smooth, monotone change of variables $t = t(x)$



Separable ODE:

$$\int^{t} \sqrt{q(t')^2 - s^2}\, dt' = c \int^{x} \sqrt{\operatorname{sech}^2(x') - r}\, dx'$$





Require (for existence of a smooth monotone $t(x)$)

$t_\pm$ correspond to $x_\pm$

Choose int. const. so that t_- corresp. to x_-

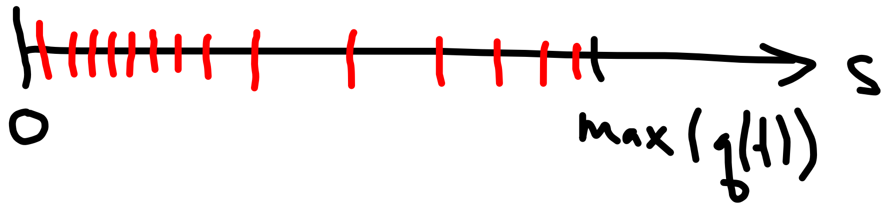
Then the condition is:

$$\int_{t_-}^{t_+} \sqrt{q_b(t')^2 - s^2}\, dt' = c \int_{x_-}^{x_+} \sqrt{\operatorname{sech}^2(x') - r^2}\, dx'$$

$$\Rightarrow \quad r = 1 - \frac{1}{\|q_b\|_1} \int_{t_-(s)}^{t_+(s)} \sqrt{q_b(t)^2 - s^2}\, dt$$

Bohr-Sommerfeld

Up to perturbation terms, this gives eigenvalues s from the sech eigenvalue r.



Local density of eigenvalues comes from the Bohr-Sommerfeld formula.

Still see local "level repulsion" but the density is <u>not</u> universal.

Various aspects: work in progress with J. Baik, K. McLaughlin

How to analyze perturbation?

① Make a "higher-order" correction to the Langer XF: result is.

$$\varepsilon \frac{d\vec{y}}{dx} = \frac{\|q\|_1}{\pi}\begin{bmatrix}0 & r(s)-s\,\mathrm{sech}(x)\\ r(s)+s\,\mathrm{sech}(x) & 0\end{bmatrix}\vec{y} - \varepsilon^2\begin{bmatrix}0 & g(x;s,\varepsilon)\\ 0 & 0\end{bmatrix}\vec{y}$$

g comes from the Langer XF and is "nice"

(2) "Solve" for $\vec{y}$ using variation of parameters:

Suppose $\underline{\underline{Y}}_0(x)$ is a fund. sol'n matrix of the $g \equiv 0$ problem.

Seek $\vec{y}$ in the form $\vec{y}(x) = \underline{\underline{Y}}_0(x)\vec{k}(x)$

$\Rightarrow$ integral eqn for $\vec{k}$.

(3) Use asymp (ε small) for $\underline{\underline{Y}}_0(x)$ to control iterates of the int. eqns.