Semiclassical Riemann-Hilbert asymptotics

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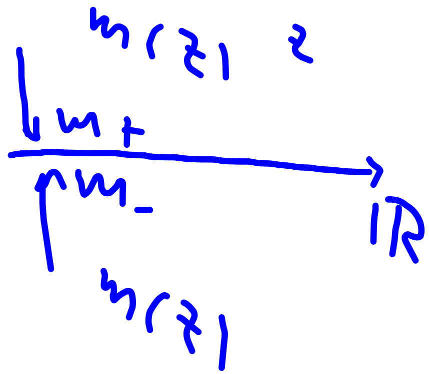
<u>RHP</u>: Find a 2×2 matrix $m(z)$, $z \in \mathbb{C}\setminus\mathbb{R}$ analytic off real axis with.

1. Normalization $m(z) \to$ Identity as $z \to \infty$

2. Jump

$$m_+ = m_- \begin{pmatrix} 1+|r|^2 & \bar{r} \\ r & 1 \end{pmatrix} \quad \text{when } z \in \mathbb{R}$$

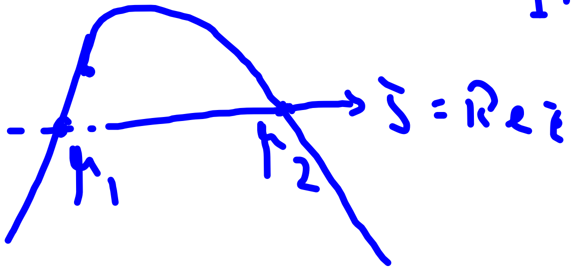
$$m_\pm = \lim_{z \to \mathbb{R}} m \quad \begin{cases} \text{from uhp} \\ \text{lhp} \end{cases}$$



Problem: Given r on the contour $\mathbb{R}$, find $m(z)$

Generally, normalization could be at any point; jump at "any" contour in $\mathbb{C}$ → general RHP.

What is $r(z)$?



graph of a real function $w(\zeta)$

$$w(\zeta) \sim \begin{cases} -k\zeta & \zeta \to +\infty \\ k\zeta & \zeta \to -\infty \end{cases}$$

INPUT.

$$r(\zeta) = \chi_{[\mu_1,\mu_2]} e^{-\frac{2i}{\varepsilon} f} \qquad \zeta \in \mathbb{R}$$

like Fourier propag. in linear PDE!

$$\boxed{f = f_0 - zx - 2z^2 t}$$

f_0 is analytic in the UHP

$$\mathrm{Im} f_{0+} = w(\zeta)\,\mathrm{sgn}(\mu_2 - \zeta)$$

Semiclassical: $\varepsilon \to 0$

$\boxed{\text{TO FIND: } m(z)}$

NL integrable system: $N(q)=0$

ex. $i\varepsilon q_t + \varepsilon^2 q_{xx} + |q|^2 q = 0$ (NLS)

$$\left\{\begin{array}{l}\partial_x \Psi = A\Psi \\ \partial_t \Psi = B\Psi\end{array}\right.$$ A, B involve $q, q_x, \ldots$

compatibility $\Leftrightarrow$ NLS

inherit propagator $e^{i(zx+2z^2t)}$

$\boxed{\Psi \text{ is almost } m}$

Linear Syst

Fourier integral

Input: $\underbrace{\text{initial data}}_{\text{FT of initial data}}$

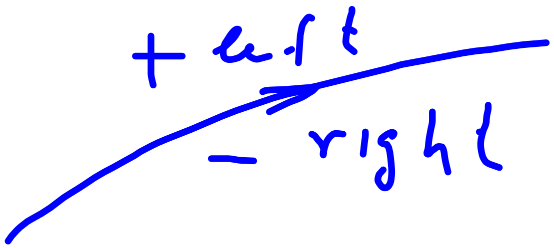
NL system

RHP

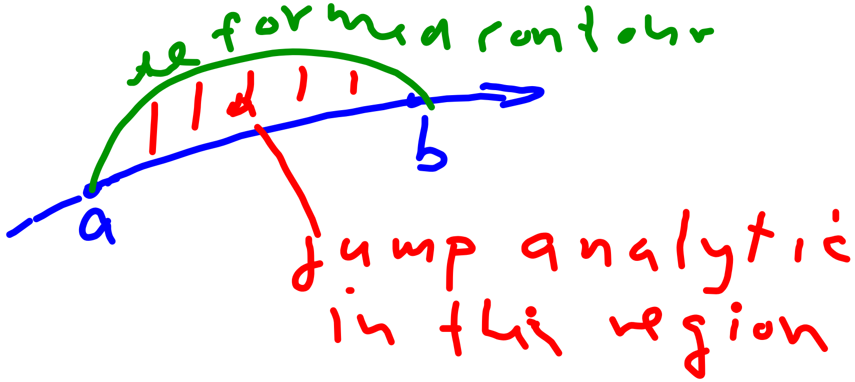
Input: scattering data of initial data.

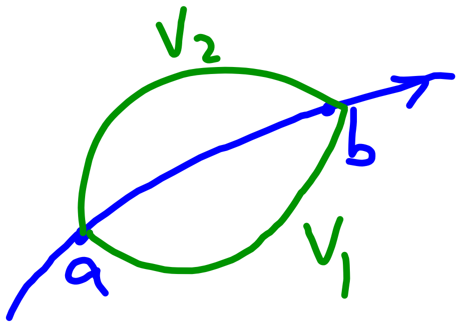
$N\{ \overset{\text{ini data}}{q(x,0)} \}$

Notation



contour deformation



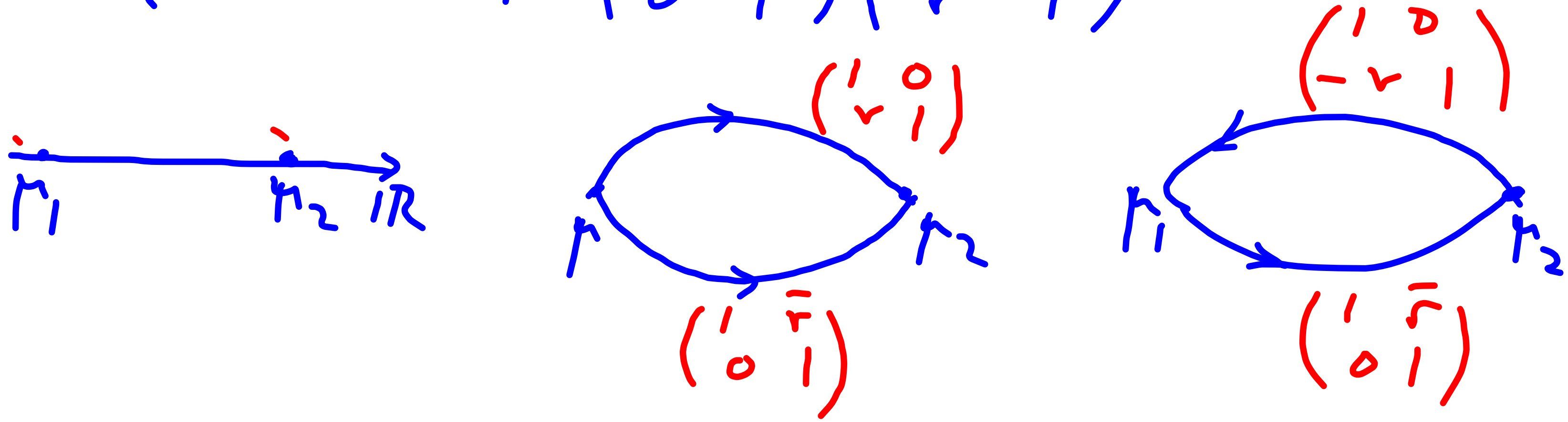


V_1, V_2 deform analytically through region of their analyticity.

factoring + deformation

Jump $V = V_1 V_2 \Rightarrow \begin{cases} \text{left factor move right} \\ \text{right factor move left} \end{cases}$

$$\begin{pmatrix} 1+|r|^2 & \bar r \\ r & 1 \end{pmatrix} = \begin{pmatrix} 1 & \bar r \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}$$



Strategy: Steepest descent for RHP

Deift, Zhou

g-function mechanism

Deift, V.., Zhou

needed when asympt. waves are fully NL.

g-funct. mech.

factoring + contour deformation leading to RHP where as $\varepsilon \to 0$

- Jumps and their inverses are bdd
- Jumps $\longrightarrow$ piecewise constant

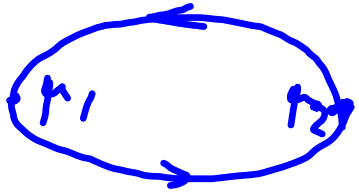
Tranf: $m(z) = m^{(1)}(z) \begin{pmatrix} e^{-2ig(z)/\varepsilon} & 0 \\ 0 & e^{2ig(z)/\varepsilon} \end{pmatrix}$

$g(z)$ analytic off the contour (to be determined) and at ∞

SYMMETRY: $m(z)\, m(\bar{z})^{*} = I$; $V(z)\, V(\bar{z})^{*} = I$

Transformed jump matrix (upper h p)

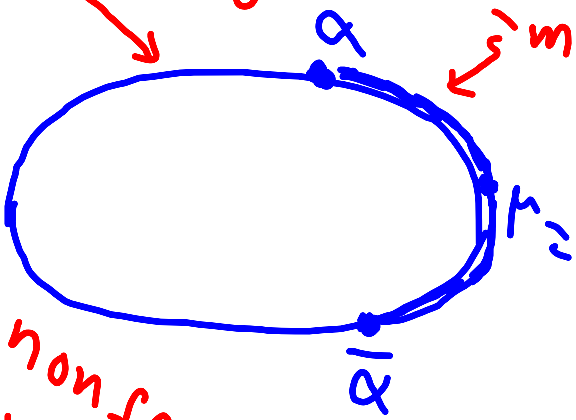
$$\begin{pmatrix} e^{\frac{2i}{\varepsilon}(g_+-g_-)} & 0 \\ -e^{\frac{2i}{\varepsilon}(g_++g_--f)} & e^{-\frac{2i}{\varepsilon}(g_+-g_-)} \end{pmatrix},$$



more factoring

$$\begin{pmatrix} a & 0 \\ -b & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & -ab^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & b^{-1} \\ -b & 0 \end{pmatrix} \begin{pmatrix} 1 & -a^{-1}b^{-1} \\ 0 & 1 \end{pmatrix}$$

complementary arc, work with nonfactored jump



main arc work with factored jump

main arc (UHP)

$g_++g_--f=0$

$\operatorname{Im} g_++g_--f<0$ both sides of contour

Compl. arc

$\operatorname{Im}(g_++g_--f)>0$

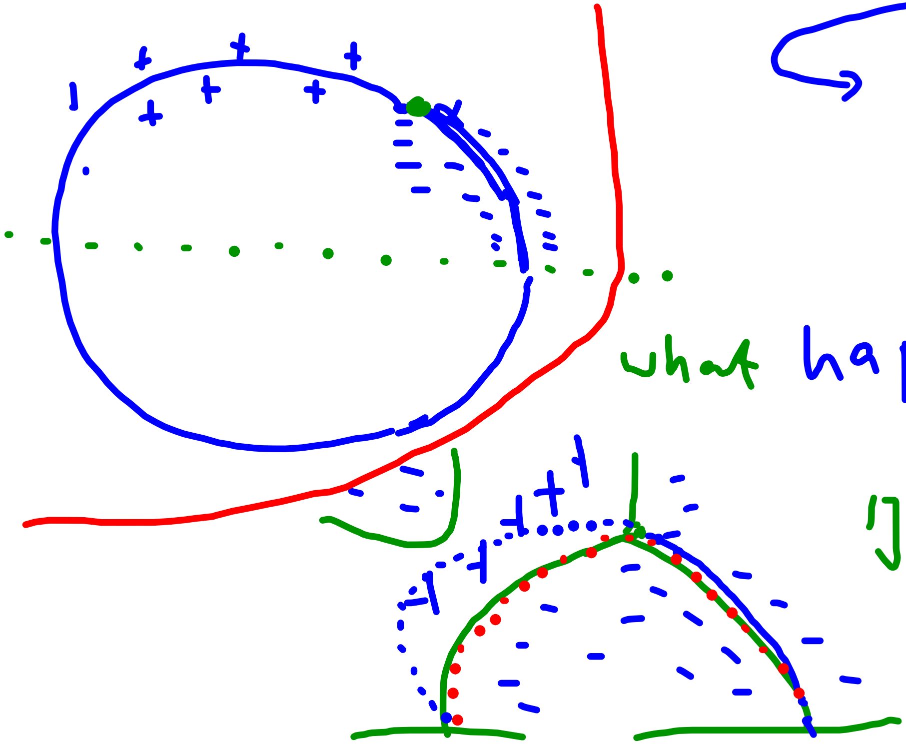
sign structure for $\operatorname{Im} g_+ + g_- - f$

(convenience: let $h = 2g - f$)

$\Rightarrow$ $\operatorname{Im} h < 0$ left and right of contour

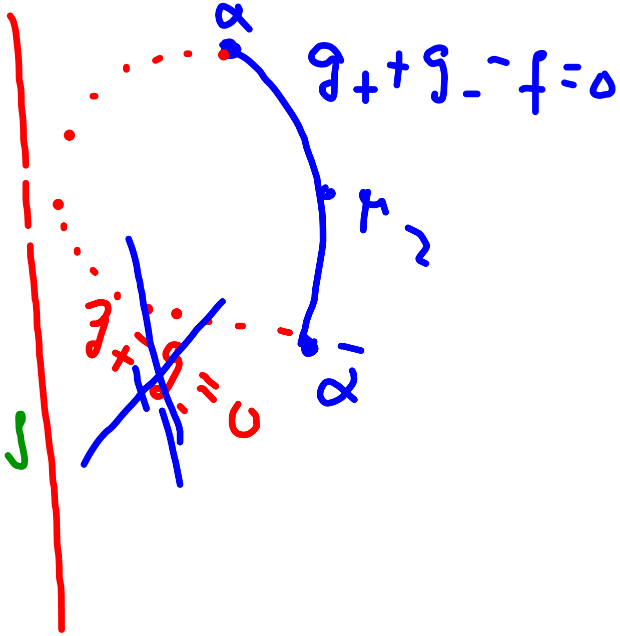
Sign profile of $\operatorname{Im} h$.

$\begin{cases} 0 & \text{main arc} \\ > 0 & \text{comp. arc} \end{cases}$



what happens

$\operatorname{Im} h = 0$ level lines

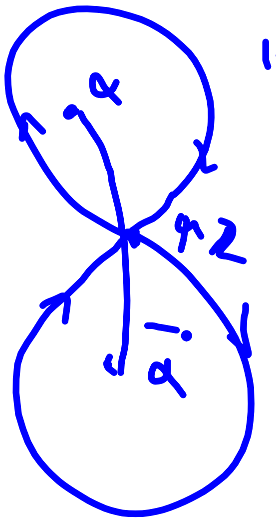


On $\cdots\cdots$ $\operatorname{Im} h = 0$

Formula for g $\qquad h = g - f$

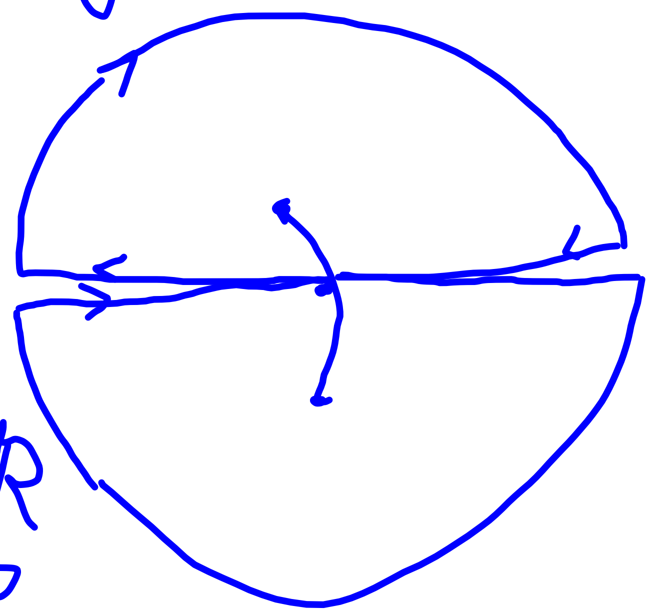
$$g'(z) = \frac{R(z)}{4\pi i} \int_\gamma \frac{f'(\zeta)}{(\zeta - z) R(\zeta)}\, d\zeta \qquad z \text{ outside } \gamma$$

$$h'(z) = \frac{R(z)}{2\pi i} \int_\gamma \frac{f'(\zeta)}{(\zeta - z) R(\zeta)}\, d\zeta \qquad z \text{ inside } \gamma$$



"fig 8" contour $= \gamma$

$\rightarrow$ deform γ



$$R(z) = \left((z-\alpha)(z-\bar{\alpha})\right)^{1/2}$$

calc on $\mathbb{R}$ and at ∞

Note: $\left(\frac{g}{R}\right)_+ - \left(\frac{g}{R}\right)_- = \frac{g_+}{R_+} - \frac{g_-}{R_-} = \frac{g_+ + g_-}{R_+} = \frac{f}{R_+}$

If we pick main arc as branch cut of $\hat{R}$ then $R_+ = -R_-$

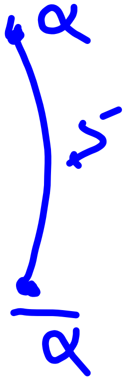
To get $\frac{g}{R}$ apply Cauchy operator as in formula.

At $z \to \infty$ $g' = O\left(\frac{1}{z^2}\right)$

Moment cond.

$$\int_\gamma \frac{f'(\zeta)}{R(\zeta)}\,d\zeta = 0 \qquad \int \frac{\zeta f'(\zeta)}{R(\zeta)}\,d\zeta = 0$$

Transformed RHP.



$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

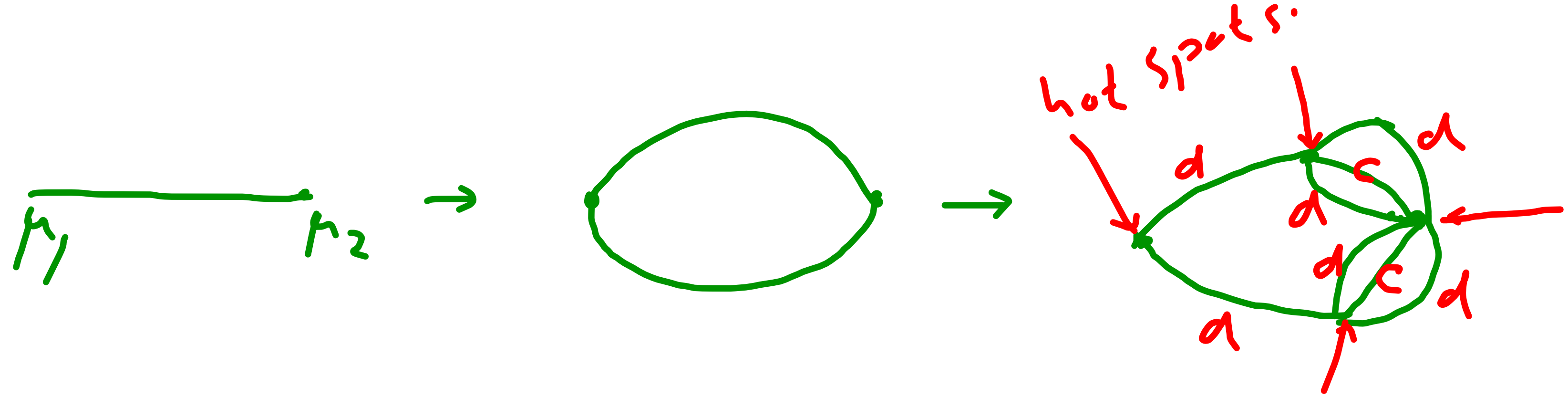
Solution:

$$m^{(1)} = \begin{pmatrix} \Sigma & -i\Delta \\ i\Delta & \Sigma \end{pmatrix}$$

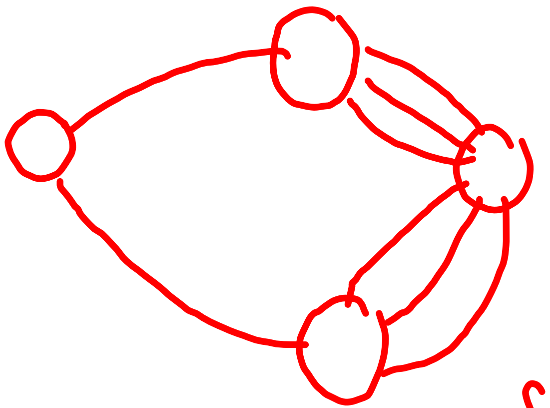
$$\Sigma = \frac{\lambda(z) + \lambda^{-1}(z)}{2}$$

$$\Delta = \frac{\lambda(z) - \lambda^{-1}(z)}{2}.$$

$$\lambda(z) = \left(\frac{z-\alpha}{z-\bar{\alpha}}\right)^{1/4}$$



Error RHP



d: decay to I

c: constant

Model problem

throw away d's

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Jump $= I + O(\varepsilon)$

Solution

$I + O(\varepsilon)$