Some examples of universality

1. CLT $\quad X_1, X_2, \ldots$ iid $\quad EX_i = 0$, $EX_i^2 = 1$ $\quad \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \xrightarrow{d} N(0,1)$

~~Donsker~~ $\quad S_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} X_i \qquad 0 \le t \le 1$

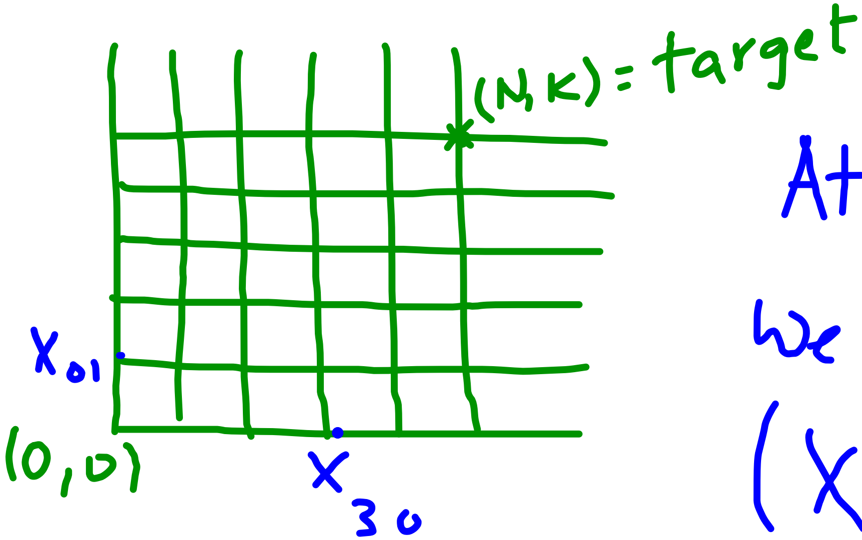
$$S_n \xrightarrow{d} BM$$

2. Wigner's semicircle law: X_{ij} iid $1 \le i, j \le n$ $\quad X_{ij} = \overline{X_{ji}}$

$\frac{1}{\sqrt{n}} ((X_{ij})) \to$ evalues $\lambda_1^{(n)}, \ldots, \lambda_n^{(n)}$ $\quad EX_{ij} = 0$, $EX_{ij}^2 = 1$

Then $\frac{1}{n} \sum_{k=1}^{n} \delta_{\lambda_k^n} \xrightarrow{P}$ W.S.L.

3. Last passage percolation:



At each vertex (i,j) we have an independent r.v. X_{ij}

$(X_{ij}$ iid$)$

For each NE going path π from $(0,0)$ to (N,K) define the time along π to be $\sum_{(i,j)\in\pi} X_{ij}$

$$L_{(N,K)} = \max_{\pi:(0,0)\nearrow(N,K)} \sum_{v\in\pi} X_v$$

Fact (Johansson): If X_{ij} are iid Geometric(q)

$\left(P(k) = q^{k-1}(1-q)\right)$

Then

$$\frac{L_{(N,\lfloor \rho N \rfloor)} - f(\star)}{\star\, N^{1/3}} \xrightarrow{d} TW_2$$

Q: Other iid random variables?

Also: Thin rectangles. $E|X_{ij}|^3 < \infty$

if $K = o(N^{\alpha})$, $\alpha < \frac{1}{7}$

$$\frac{L(N,K) - (\star)}{K^{-1/6} N^{1/2}} \xrightarrow{d} TW_2$$

4. <u>Circular law:</u>

X_{ij} iid $\quad E X_{ij} = 0$

$E X_{ij}^2 = 1$

$\lambda_1^n, \ldots, \lambda_n^n \to$ evalues of $\frac{1}{\sqrt{n}} ((X_{ij}))_{i,j \le n}$

$$\frac{1}{n} \sum_{k=1}^{n} \delta_{\lambda_k^n} \xrightarrow{P} \text{Uniform measure on the unit disk in } \mathbb{C}$$

Lindeberg (1929)

. Chatterjee (2004)

Rotar; Mossel et al.

Problem of universality:

$X = (X_1, X_2, \ldots, X_n)$ X_i independent $\mathbb{R}$-valued rvs.

$Y = (Y_1, \ldots, Y_n)$ Y_i independent $\mathbb{R}$-valued rvs

$$E X_i = E Y_i \, ; \, E X_i^2 = E Y_i^2$$

$T : \mathbb{R}^n \longrightarrow \mathbb{R}$ a fn

$U = T(X) \qquad V = T(Y)$

Want to say U and V are close in law.

$\iff \quad E[g(U)] \approx E[g(V)]$ for any $g: \mathbb{R} \to \mathbb{R}$ (nice)

<u>Assumptions:</u> 1) T is 3-times coordinatewise differe.

$T: \mathbb{R}^n \to \mathbb{R}$ 2) g is 3-times differentiable

$g: \mathbb{R} \to \mathbb{R}$

$h = g \circ T$ Construct X, Y on the same prob. space and ind.

$h(X_1, \ldots, X_n) - h(Y_1, \ldots, Y_n)$

$= \sum_{i=1}^{n} h(X_1, \ldots, X_{i-1}, X_i, Y_{i+1}, \ldots Y_n) - h(X_1 \ldots, X_{i-1}, Y_i, Y_{i+1} \ldots Y_n)$

$$h(X_1, \ldots, X_{i-1}, X_i, Y_{i+1} \ldots, Y_n) - h(\overbrace{X_1, \ldots, X_{i-1}, 0, Y_{i+1}, \ldots, Y_n}^{W_i})$$

$= \partial_i h(W_i) X_i + \partial_i^2 h(W_i) \frac{X_i^2}{2} + R_i$

$h(X_1, \ldots, X_{i-1}, Y_i, \ldots Y_n) - h(W_i)$

$= \partial_i h(W_i) Y_i + \partial_i^2 h(W_i) \frac{Y_i^2}{2} + S_i$

$$E\left[h(x_1,\dots,x_{i-1},x_i,y_{i+1},\dots y_n) - h(x_1,\dots,x_{i-1},y_i,\dots y_n)\right]$$

$$= E\left[\partial_i h(w_i)\,(x_i - y_i)\right] + E\left[\frac{1}{2}(x_i^2 - y_i^2)\,\partial_i^2 h(w_i)\right] + E\left[R_i - S_i\right]$$

$$= E\left[R_i - S_i\right]$$

$$\leq E\left[|R_i| + |S_i|\right]$$

$$R_i = h(\ldots) - h(w_i) - \partial_i h(w_i) x_i - \partial_i^2 h(w_i) \frac{x_i^2}{2}$$

1) $$|R_i| = \left| \frac{1}{6} \partial_i^3 h(\tilde{w}_i)\, x_i^3 \right| \leq \frac{1}{6} \|\partial_i^3 h\|_\infty \cdot |x_i|^3$$

2) $$|R_i| = \left| \frac{1}{2} \partial_i^2 h(\tilde{\tilde{w}}_i)\, x_i^2 - \frac{1}{2} \partial_i^2 h(w_i)\, x_i^2 \right|$$

$$\leq \|\partial_i^2 h\|_\infty \cdot |x_i|^2$$

$$h = g \circ T$$

$$\partial_i h' = g' \circ T \cdot \partial_i T$$

$$h = g \circ T$$

$$\partial_i h = (g' \circ T)\partial_i T$$

$$\partial_i^2 h = (g'' \circ T)(\partial_i T)^2 + (g' \circ T)\partial_i^2 T$$

$$\|\partial_i^2 h\|_\infty \leq \left(\|g''\|_\infty + \|g'\|_\infty\right)\lambda_2(T) = c_1(g)\lambda_2(T)$$

$$\lambda_2(T) = \max\{|\partial_i T|_x^2, |\partial_i^2 T|_x ; i, x\}$$

Similarly $\|\partial_i^3 h\|_\infty \leq c_2(g)\cdot\lambda_3(T)$

$$\lambda_3(T) = \max\{(\partial_i T)^3, (\partial_i^2 T)^{3/2}, |\partial_i^3 T|\}$$

$$c_2(g) = 10\left(\|g'''\|_\infty + \|g''\|_\infty + \|g'\|_\infty\right)$$

$$\left|E\left[h(X)-h(Y)\right]\right| \le \sum_{i=1}^{n} E\left[|R_i| + |S_i|\right]$$

$$\le c_1(g)\,\lambda_2(T)\sum_{i=1}^{n} E\left[X_i^2\,\mathbb{1}(|X_i|>k)\right] + E\left[Y_i^2\,\mathbb{1}(|Y_i|>k)\right]$$

$$+\, c_2(g)\,\lambda_3(T)\sum_{i=1}^{n} E\left[|X_i|^3\,\mathbb{1}(|X_i|\le k)\right] + E\left[|Y_i|^3\,\mathbb{1}(|Y_i|\le k)\right]$$

$$E\left[|X_i|^3\,\mathbb{1}(|X_i|\le k)\right] \le k\,E\left[|X_i|^2\right]$$

1) CLT: $X = (X_1, \ldots, X_n)$ $\quad U = T(X)$

$Y = (Y_1, \ldots, Y_n)$ $\quad V = T(Y)$ $\quad Y_i \sim N(0,1)$

$$T(x_1, \ldots, x_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i \qquad \lambda_2(T) = \frac{1}{n}$$

$$\partial_i T = \frac{1}{\sqrt{n}} \quad \partial_i^2 T \equiv 0 \qquad \lambda_3(T) = \frac{1}{n^{3/2}}$$

$$\left| E[g(U)] - E[g(V)] \right| \leq C_1(g) \frac{1}{n} \sum_{i=1}^{n} E\left[X_i^2 \mathbb{1}(|X_i| > \epsilon\sqrt{n})\right]$$

$$+ C_2(g) \frac{1}{n^{3/2}} \cdot \epsilon\sqrt{n} \cdot n$$

2) W.S.L: X_{ij} iid (realvalued) $i,j \leq n$ $X_{ij} = X_{ji}$

$\frac{X}{\sqrt{N}}$ has evalues $\lambda_1^n, \dots, \lambda_n^n$

we consider $\frac{1}{N} \sum_{k=1}^{n} \delta_{\lambda_k^n}$

Stieltjes transform:

Fix $z \in \mathbb{H}$ Consider

$$T(X) = \frac{1}{N} \sum_{k=1}^{N} \frac{1}{\lambda_k - z} = \frac{1}{N} \operatorname{Tr}\left[\left(\frac{X}{\sqrt{N}} - zI \right)^{-1} \right]$$

$$T(X) = \frac{1}{N} \operatorname{Tr}\left[\left(\frac{X}{\sqrt{N}} - zI\right)^{-1}\right] = \frac{1}{N}\operatorname{Tr}\left(A^{-1}\right)$$

$$A^{-1} = B$$

$$BA = I \qquad\qquad A = \frac{X}{\sqrt{N}} - zI$$

$$\partial_{ij}B \cdot A + B \cdot \partial_{ij}A = 0$$

$$\partial_{ij}B = -B\,\partial_{ij}A\,B \qquad\qquad \partial_{ij}A = \frac{1}{\sqrt{N}}\left(\Delta_{ij} + \Delta_{ji}\right)$$

$$\partial_{ij}^2 B = 2B\,\partial_{ij}A\,B\,\partial_{ij}A\,B - B\,\partial_{ij}^2 A\,B \qquad\qquad \partial_{ij}^2 A = 0$$

$$\partial_{ij}^3 B = \ldots$$

We can calculate:

$$\lambda_2(T) \le \frac{4}{N^2} \cdot \max\left(\frac{1}{v^4}, \frac{1}{v^3} \right)$$

$$\lambda_3(T) \le \frac{12}{N^{5/2}} \cdot \max\left(\frac{1}{v^4}, \frac{1}{v^6} \right)$$

$v := \operatorname{Im}(z)$

$$E\, g(T(X)) - E\, g(T(Y)) \le \frac{c_1(g)}{N^2} \sum_{ij} E X_{ij}^2 \mathbb{1}(|X_{ij}| > K) + \frac{c_2(g)}{N^{3/2}} \sum_{ij} E |X_{ij}|^3 \mathbb{1}(\quad < K)$$

$K = \epsilon\sqrt{N}$

$$E\left[|X_{ij}|^3 \mathbb{1}(|X_{ij}| < \epsilon\sqrt{N})\right] \leq \epsilon\sqrt{N} \cdot E\left(|X_{ij}|^2\right)$$

Pastur's condition:

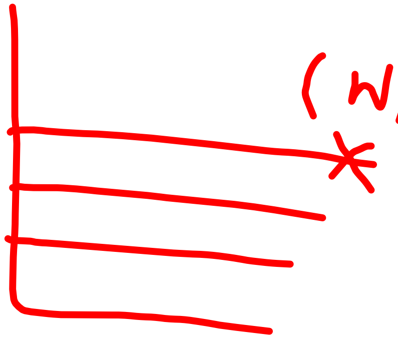
If

$$\frac{1}{N^2}\sum_{i,j=1}^{n} E\left[X_{ij}^2 \mathbb{1}(X_{ij} > \epsilon\sqrt{N})\right] \longrightarrow 0$$

$\forall \epsilon > 0$

$\Longrightarrow$ S.S.

Last Passage percolation



(N,K)

$$\mathcal{T} := \{\pi : (0,0) \nearrow (N,K)\}$$

$$T(X) = \max_{\pi \in \mathcal{T}} \sum_{v \in \pi} X_v$$

Trick:

$$T(x) = \boxed{\max_{\pi \in \mathcal{F}} \pi(x)}$$

$$e^{\beta \max_\pi \pi(x)} \le \sum_{\pi \in \mathcal{F}} e^{\beta \pi(x)} \le \#(\mathcal{F}) \cdot e^{\beta \max_\pi \pi(x)}$$

$$T(x) \le \frac{1}{\beta} \log \sum_\pi e^{\beta \pi(x)} \le \pi(x) + \frac{\log(\#\mathcal{F})}{\beta}$$

$$\frac{1}{\beta} \log \sum_\pi e^{\beta \pi(x)} = T_\beta(x)$$

$$\left| E\, g(T(X)) - E\, g(T(Y)) \right|$$

$$\frac{(\max_\pi \pi(x) - \cancel{T})}{K^{-1/6} N^{1/2}}$$

$$\leq \left| E\left[g(T_\beta(X)) \right] - E\left[g(T_\beta(Y)) \right] \right|$$

$$+ \frac{2}{\beta} \log(\#7) \, \|g'\|_\infty$$

$$\partial_v T_\beta = \frac{1}{\beta} \frac{\sum e^{\beta\pi(x)} \beta\, \partial_v \pi(x)}{\sum_\pi e^{\beta\pi(x)}}$$

$$T_\beta(X) = \frac{1}{\beta} \log \sum_{\pi : (0,0) \nearrow (N,K)} e^{\beta\pi(x)}$$

$$\partial_v T_\beta$$

$$\partial_v^2 T_\beta$$

$$\vdots$$

$$\lambda_3(T_\beta) \leq \frac{K^{1/6}}{N^{3/2}}$$