1. ZS
$$\begin{cases} \partial_x W = -\dfrac{i}{\varepsilon}\underbrace{\begin{pmatrix} z & q(x,t,\varepsilon) \\ \bar{q}(x,t,\varepsilon) & -z \end{pmatrix}}_{A} W \\[2ex] \partial_t W = \dfrac{i}{\varepsilon} B W \end{cases}$$

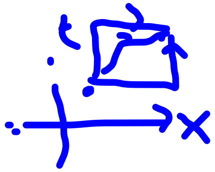
$W = W(x,t,z)$

W: 2×2 matrix

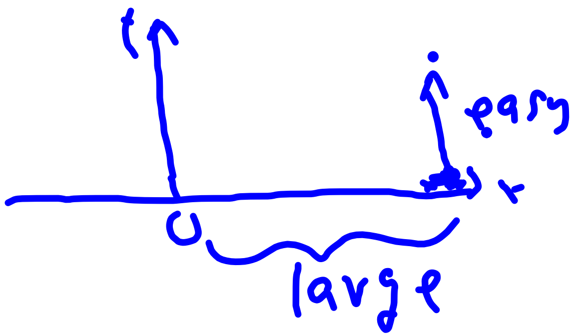
ε: paramet.

B: "like A" depends on z, q, q_x

2. $$\underbrace{W_{xt} = W_{tx}}_{\text{compatibility}} \iff \underbrace{i\varepsilon q_t + \varepsilon^2 q_{xx} + |q|^2 q = 0}_{\substack{\text{Focusing NLS} \\ \text{(NLS)}}}$$



3 Solution of NLS given $q(x,0,\varepsilon)$



calculate

1. asymptotic limit $\bar{W}_{as}(x,t,z,\varepsilon)$ as $x \to \pm\infty$

(for $q(x,0,\varepsilon) \to 0$ as $x \to \pm\infty$ and for z's with $\bar{W}$ bounded)

1. $z \in \mathbb{R}$ (scattering solutions W)

wave nature $\notin L_2$

$e^{-i\omega \frac{t}{\varepsilon}} W(x)$

$e^{\frac{i}{\varepsilon}(\pm zx - \omega t)}$

$$W_{as} = \begin{cases} \overset{\text{trans. coef.}}{T(z)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \underset{\leftarrow \text{trans}}{e^{-\frac{iz}{\varepsilon}x}} & x \to -\infty \\ \underset{\text{incid.}}{\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-\frac{iz}{\varepsilon}x}} + \underset{\text{refl. coef.}}{r(z)} \underset{\text{refl.}}{\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{\frac{iz}{\varepsilon}x}} & \text{as } x \to +\infty \end{cases}$$

column of W

W_{as} specified given $T(z)$ and $r(z)$

actually $r(z)$ suffices

$$|T|^2 - |r|^2 = 1$$

2. Bound solutions or Bound States.
or Localized solutions.

pairs $z_j, \bar{z}_j$ ZS has L^2 solution W_j.

norming constants. c_j

asympt. beh. of W_j as $x \to +\infty$

$\|W_j\|_{L_2} = 1.$

Evolution

$$r(z,t) = r(z,0)\, e^{2\frac{i}{\epsilon} z^2 t}.$$

$$z_j(t) = z_j(0)\,; \quad c_j(t) = c_j(0)\, e^{2\frac{i}{\epsilon} z_j^2 t}$$

To find $q(x,t,\varepsilon)$ we go to map

$$W_{as}(x,t,z,\varepsilon) \longrightarrow W(x,t,z,\varepsilon)$$

inverse scattering map

Better.

$$r(z) \; ; \; \{(z_j, c_j)\} \longrightarrow q(x,t)$$

Solution of ISP (inv. scat. problem)
through RH problem.

- x, t parameters
- work in z plane

$\Big\}$ $W = W(z)$ — x, t constants.

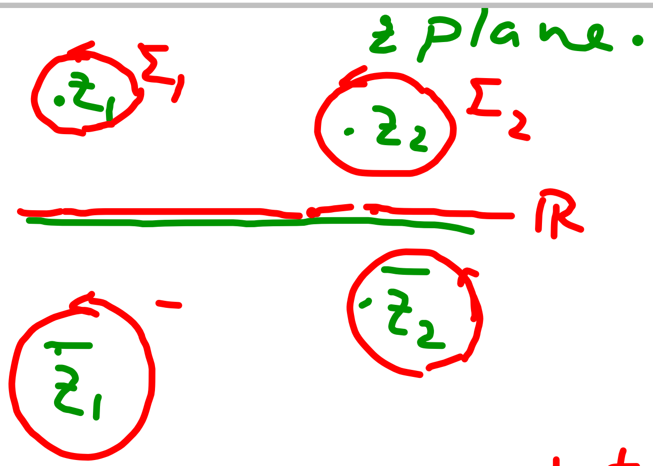
Change $m = W \begin{pmatrix} e^{iz x} & 0 \\ 0 & e^{-iz x} \end{pmatrix}$

: $m(z) \longrightarrow I = \text{Identity} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ as $z \to \infty$

$$m = I + \frac{1}{z}\tilde{m} + O\left(\frac{1}{z^2}\right) \quad \text{as } z \to \infty$$

$$q(x,t,\varepsilon) = -2\tilde{m}_{\underline{12}}(x,t,\varepsilon)$$

$\underline{12}$ ← indicates entry.



Seek $m(z)$ analytic everywhere except on red contours and $m \to I$ as $z \to \infty$

indices
+ limit from left
− " right

$$m_+ = m_- \begin{pmatrix} 1+|R|^2 & \bar{R} \\ R & 1 \end{pmatrix}, \quad z \in \mathbb{R}$$

$$m_+ = m_- \begin{pmatrix} 1 & 0 \\ \frac{c_j}{z - z_j} & 1 \end{pmatrix} \quad z \in \Sigma_j$$

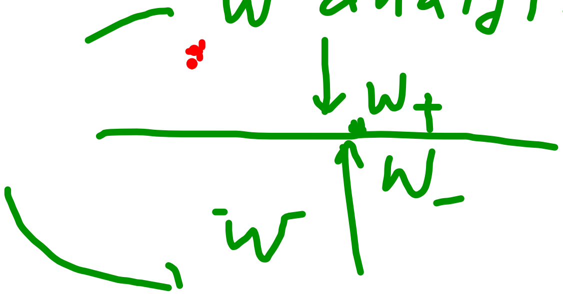
Approach as $\varepsilon \to 0$

- peel off "stuff" from m
- arrive to RHP for $m^{(error)} \approx I$ and can be estimated.

Where does RHP come from.

(for simplicity assume no bound states)

W analytic in z.



On $\mathbb{R}$, W_+, W_- are fundamental solutions of ZS. Thus.

$$W_+ = W_- \cdot J$$

J: 2x2 matrix independent of x but dependent on z, specifically on $r(z)$

To define W on the upper half plane, we use scattering solutions for each column chosen to have analytic ext. in uhp.

Symmetry for W in lower half plane

$R = r_0(z)\, e^{\frac{2i}{\varepsilon} z x + \frac{4i}{\varepsilon} z^2 t}$

at $t = 0$

Asymptotics $\varepsilon \to 0$

Analogy

<u>Linear PDE</u>	<u>NL PDE (integrable)</u>
Fourier Integral	Riemann Hilbert Pr. (RHP)
Laplace | stat. phase / Steepest descent (SD)	steepest descent.
"Main contributors": points	"Main contributors": contours.

ORIGINAL THEORY : KdV eqn.

Lin. op.

$$\begin{cases} -\varepsilon^2 \partial_{xx} w + q(x,t) w = \lambda w \\ \text{evolution equation} \end{cases}$$

compat : KdV

$$q_t - 6 q q_x + \varepsilon^2 q_{xxx} = 0$$

* refl., transm.; bound states.

Formula for q in pure soliton case $(r(z) \equiv 0)$

$$q(x,t,c) = -2\varepsilon^2 \partial_{xx} \ln \tau(x,t,c)$$

τ is a sum of integrals of form.

$$\int \cdots \int e^{\frac{1}{\varepsilon}\left[\sum_{i=1}^{n} V(\lambda_i; x,t)\right] + \sum_{j=1}^{n} \sum_{\substack{i=1 \\ i \neq j}}^{n} \ln|\lambda_i - \lambda_j|} d\lambda_1 \ldots d\lambda_n$$

$\bar{V}(\lambda)$: Scattering data. $V = V_0(\lambda) + 2\lambda^{1/2} x + 4\lambda^{3/2} t$