Non-Hermitian random matrices.

$A = \frac{1}{\sqrt{N}} ((X_{ij}))_{ij \le N}$ $\quad X_{ij}$ iid $\mathbb{C}$-valued rvs.

Eg $X_{ij} \sim$ iid $\mathbb{C}N(0,1)$ $\left(\text{pdf} = \frac{e^{-|z|^2}}{\pi} \right)$

Then $(\lambda_1^A, \dots, \lambda_N^A)$ has pdf $\quad \lambda_i \in \mathbb{C}$

$$\frac{1}{Z_N} \cdot e^{-N \sum_{k=1}^{N} |\lambda_k|^2} \cdot \prod_{i<j} |\lambda_i - \lambda_j|^2$$

1. Expected measure of evalues, ie $E[L_N] = E\left[\frac{1}{N}\sum_{k=1}^{N}\delta_{\lambda_k^A}\right]$

In fact $\longrightarrow \text{Unif}(\mathbb{D})$ unit disk

$L_N \xrightarrow{P} \text{unif}(\mathbb{D})$ (Silverstein)

2. $\{N\lambda_1^A, \dots, N\lambda_N^A\} \longrightarrow$ a determinantal p.p. on $\mathbb{C}$ with kernel

$$K(z,w) = \frac{e^{-\frac{1}{2}|z|^2 - \frac{1}{2}|w|^2 + z\bar{w}}}{\pi}$$

Q: What if X_{ij} are iid

$EX_{ij} = 0 \quad E|X_{ij}|^2 = 1$

Expect (1) to hold.

Circular law: For any X_{ij} iid $\begin{pmatrix} \text{mean}=0 \\ \text{var}=1 \end{pmatrix}$

$$L_N \xrightarrow{P} \text{Unif}(\mathbb{D})$$

[Bai]: Under assumptions such as bounded densities ...

What's the difficulty compared to Semicircle?

Lack of SYMBOLIC CALCULUS

Hermitian methods

1) Moments: $\frac{1}{N}\mathrm{Tr}(A^k) = \frac{1}{N}\sum_{j=1}^{N}(\lambda_j^A)^k = \int x^k \, dL_N(x)$

Non Hermitian: For a $\mathbb{C}$-valued rv ξ need $E\left[\xi^k \bar{\xi}^\ell\right]$

2) Stieltjes transform: $\frac{1}{N}\mathrm{Tr}(zI-A)^{-1} = \frac{1}{N}\sum_{k=1}^{N}\frac{1}{z-\lambda_k}$

Non-Hermitian:

Unif($\mathbb{D}$)

δ_0 : have the same S.T. on $\mathbb{D}^c$

Way out: Fact: f – a complex analytic function

$$\frac{1}{2\pi}\Delta \log|f(z)| = \sum_{\alpha\in f^{-1}\{0\}} \delta_\alpha$$

We want zeros of $f(z) = \det(zI - A)$

$$L_N = \frac{1}{N}\cdot\frac{1}{4\pi}\Delta_z \log\det\left[(zI-A)(zI-A)^*\right]$$

$$= \frac{1}{4\pi N}\cdot\Delta_z \int_0^\infty \log(x)\, \nu_N(dx, z)$$

$$dL_N^{(z)} = \frac{1}{4\pi} \cdot \Delta_z \int_0^\infty \log(x)\, \nu_N(dx, z)$$

$$\nu_N(\cdot, z) = \text{E.S.D. of } \underbrace{(zI-A)(zI-A)^*}_{\text{"}H_z\text{"}}$$

$$\nu_N = \frac{1}{N}\sum_{k=1}^{N} \delta_{\lambda_k^{H_z}}$$

Girko's approach (also Bai's)

1) For each z, H_z is a Hermitian random matrix

$$\nu_N(\cdot, z) \xrightarrow{P} \nu(\cdot, z) \text{ sufficiently well}$$

2) Justify: $L_N \xrightarrow{P} \frac{1}{4\pi} \Delta_z \int_0^\infty \log(x)\, \nu(dx, z)$

1) $\| \nu_N(\cdot, z) - \nu(\cdot, z) \|_{TV} \leq \frac{c}{N^\alpha}$ $(\alpha > 0)$

'OKAY' $\longrightarrow$ 1) Without crazy assumptions
2) Perhaps we can simplify this part

S.T of $\nu_N(\cdot, z) \longrightarrow$ S.T of $\nu(\cdot, z)$

which is a root of

$$0 = \Delta_\xi^3 + 2\Delta_\xi^2 + \frac{\xi + 1 - |z|^2}{\xi}\Delta + \frac{1}{\xi}$$

2) $$\int_{\frac{1}{M} < |z| < M} \left| \int_0^{\epsilon_N} \log(x)\, \nu_N(dx, z) \right| dz \longrightarrow 0$$

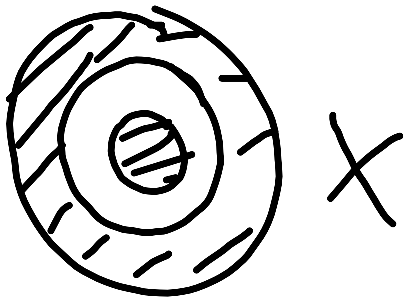
as $N \to \infty$

$\epsilon_N = e^{-N^\alpha}$

Other non-H. models $e^{-\mathrm{Tr}(V(A^*A))}$

<ins>Single-ring conj:</ins> $L_N \xrightarrow{P} \mu$

s.t. Support(μ) is connected.



<ins>Technical issue:</ins> For a specific small α

$$0 \overset{?}{\longleftarrow} N^{\alpha} P\{\exists \text{ an e.val of } A \text{ in } D(\omega, e^{-N^{\alpha}})\}$$

Another Case: $A \quad B$ ind A_{ij} B_{ij} iid $\in N(0,1)$.

evals of $A^{-1}B$ = zeros of $|zA - B|$

rotation invariant on S^2