## Circular ensembles

CUE: $U \sim \mathrm{Haar}(\mathcal{U}_n)$

$\lambda_1, \dots, \lambda_n$ = evalues of $U$

Haar: $\mu(VA) = \mu(A) = \mu(AV)$

$V$ unitary, $A \subset \mathcal{U}(n)$, $\mu(\mathcal{U}_n) = 1$

$-iU^* dU \to$ left and right invariant

$$-i(V_0U)^* d(V_0U) = -iU^* V_0^* V_0 dU = -iU^* dU$$

$dH := -iU^* dU$ is Hermitian

Then $d\mu(U) = \prod_{k=1}^{n} dm_{\mathbb{R}}(H_{kk}) \prod_{j<k} dm_{\mathbb{C}}(H_{jk})$

Fact: If $U \sim \mu$ then $\lambda_1, \dots, \lambda_n$ have pdf $C_N \prod_{j<k} |\lambda_j - \lambda_k|^2$ for $\lambda_j \in S^1$

Idea of proof: $U = V \Delta V^*$, $V$ unitary, $\Delta$ diagonal

General potential: $U \sim e^{-\mathrm{Tr}(V(U))} d\mu(U)$

$V: \mathcal{U}_n \to \mathbb{R}$ a potential

Since $\mathrm{Tr}(V(U)) = \sum_{k=1}^{n} V(\lambda_k)$

In this case $(\lambda_1, \dots, \lambda_n)$ has pdf

$$e^{-\sum_{k=1}^{n} V(\lambda_k)} \cdot \prod_{j<k} |\lambda_j - \lambda_k|^2$$

Remark: $\mathrm{pdf} = \exp\{ \sum_{k=1}^{n} V(\lambda_k) + \sum_{j<k} \log|\lambda_j - \lambda_k|^2 \}$

$$n^2 \int V \, dL_N + \frac{N^2}{2} \int \log|z-w| \, dL_N(z) \, dL_N(w)$$

pdf $e^{-\sum V(\lambda_k)} \prod_{j<k} |\lambda_j - \lambda_k|^2$

Can compute all correlation functions.

How? $\prod_{j<k} (\lambda_j - \lambda_k) = \det \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ \vdots \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{pmatrix}$

$$\text{pdf} = \det\left(\left( K(\lambda_i, \lambda_j) \right)\right)_{ij \le n}$$

$$K(x,y) = \left[ \sum_{k=0}^{n-1} \varphi_k(x) \overline{\varphi_k(y)} \right] e^{-\frac{V(x)}{2} - \frac{V(y)}{2}}$$

$\varphi_k$ – orthonormalize $1, z, z^2, \ldots z^{n-1}$

w.r.t. $e^{-V(\theta)} \frac{d\theta}{2\pi}$

Fact: $\int \det\left(\left( K(x_i, x_j) \right)\right)_{ij \le n} dx_n = \det\left(\left( K(x_i, x_j) \right)\right)_{n-1}$

and you can continue.

Eg $K(x,x)$ = density of eigenvalues.

$K(x,x)K(y,y) - K(x,y)K(y,x)$ – 2 pt corr.

$$\tilde{K}_n(x,y) = \sum_{k=0}^{n-1} \varphi_k(x) \overline{\varphi_k(y)}$$

$$= \frac{\varphi_n^*(x) \overline{\varphi_n^*(y)} - \overline{\varphi_n(y)} \varphi_n(x)}{1 - x\bar{y}}$$

$h(x) = c_0 + c_1 x + \ldots + c_n x^n$

$h^*(x) = \bar{c}_0 x^n + \bar{c}_1 x^{n-1} + \ldots + \bar{c}_n$

$1 - x\bar{y} = \frac{1}{y}(y - x)$

$$\int K_n(x,y)\, \varphi_m(y)\, dy = \begin{cases} \varphi_m(x) & \text{if } m \le n-1 \\ 0 \end{cases}$$

$$\varphi_m(y) = \varphi_m(x) + (y - x)\, \psi_{m,x}(y)$$

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# Orthogonal Polynomials on the Unit Circle

$\phi(\theta)$, $\theta \in S^1$ positive weight.

inner product: $\langle f, g \rangle_\phi := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\overline{g(\theta)}\,\phi(\theta)\,d\theta$

Apply Gram-Schmidt to $\{1, z, z^2, \dots\}$

$$= \frac{1}{2\pi i} \oint_\Sigma f(\arg(z))\,\overline{g(\arg(z))}\,\phi(\arg(z))\cdot\frac{dz}{z}$$

$\Sigma$ = unit circle

$\Rightarrow$ get a sequence of orthonormal polynomials $p_n(z) = \gamma_n z^n + \dots$

$\gamma_n > 0$

$\langle p_n(e^{i\theta}), p_m(e^{i\theta})\rangle_\phi = \delta_{mn}$.

$\pi_n(z) := \frac{1}{\gamma_n} p_n(z) = z^n + \dots$

Simplest case: $\phi(\theta) \equiv 1$.
$\pi_n(z) = z^n$
reproducing kernel is the Dirichlet kernel from Fourier Series.

Assume: $\phi$ satisfies a Hölder condition

## A tool for large-$n$ asymptotics: Riemann-Hilbert Problem.

Seek a $2\times 2$ matrix-valued fn of $z \in \mathbb{C}$, $\underline{\underline{M}}^n(z)$, with the following properties:

① $M^n(z)$ is analytic except at $|z|=1$ with continuous bdry values $M^n_\pm(z)$, $|z|=1$


satisfying $M^n_+(z) = M^n_-(z)\begin{bmatrix} 1 & \phi(\theta)e^{-in\theta} \\ 0 & 1 \end{bmatrix}$, $z = e^{i\theta}$

② $M^n(z)\begin{bmatrix} z^{-n} & 0 \\ 0 & z^n \end{bmatrix} \to \mathbb{I}$ as $z \to \infty$.

$\phi$ Hölder with exponent $\nu \in (0,1]$:

$$|\phi(\theta_1) - \phi(\theta_2)| \le K|\theta_1 - \theta_2|^\nu$$

Fact: for $n \ge 1$

$$M^n(z) = \begin{bmatrix} \pi_n(z) & \frac{1}{2\pi i}\oint \frac{\pi_n(s)s^{-n}}{s-z}\phi(\arg(s))\,ds \\ -\gamma_{n-1}^2 z^{n-1}\,\overline{\pi_{n-1}(1/\bar{z})} & \text{Same operator applied to} \end{bmatrix}$$

Baik-Deift-Johansson '99
Fokas-Its-Kitaev '91

Idea of proof:

Consider the $(1,1)$ entry in the jump condition.

$M_{11+}(z) = M_{11-}(z)$, $z = e^{i\theta}$. Cts bdry values $\Rightarrow$

$M_{11}(z)$ is entire in $z$.

(1,1) entry: Normalization: $M_{11}(z)z^{-n} \to 1$ as $z \to \infty$.

$\Rightarrow M_{11}(z)$ is a monic poly of degree $n$.

Now look at $(1,2)$ entry of jump condition:

$$M^n_{12+}(z) = M^n_{12-}(z) + M^n_{11}(z)e^{-in\theta}\phi(\theta).$$

$$\Rightarrow M^n_{12}(z) = \frac{1}{2\pi i}\oint \frac{M^n_{11}(s)s^{-n}\phi(\arg(s))}{s-z}\,ds + \underbrace{e(z)}_{\text{entire.}}$$

Apply normalization cond.

$$M_{12}(z)z^n \longrightarrow 0 \text{ as } z\to\infty.$$

$$M_{12}(z) = \underbrace{\text{Cauchy integral}}_{\text{decays as } \frac{1}{z},\ z\to\infty.} + \underbrace{e(z)}_{\text{entire}}$$

$\Rightarrow e(z)$ must go to zero as $z\to\infty \Rightarrow e(z) \underset{\uparrow}{=} 0$.
Liouville.

We require more decay of the Cauchy int.
Expand $\frac{1}{s-z}$ in geometric series, integrate term by term.
We learn that

$$\langle M^n_{11}(z), z^k\rangle_\phi = 0 \text{ for } k=0,\dots,n-1. \Rightarrow M^n_{11}(z) = \pi_n(z).$$

<u>Asymptotics</u>. $\phi(\theta) = e^{-V(\theta)}$

Assume: $V(\theta)$ extends from the circle to an analytic fn of $z$, say $\tilde V(z)$. e.g. $V(\theta) = \cos(\theta)$

$$= \tfrac12\left(e^{i\theta} + e^{-i\theta}\right)$$
$$= \tfrac12\left(z + \tfrac1z\right) = \tilde V(z).$$

Note: target is a RHP for a near-identity matrix.
normalization $\begin{pmatrix} z^n & 0 \\ 0 & z^{-n}\end{pmatrix}$ of $M^n(z)$ is an obstruction.
Remove it!

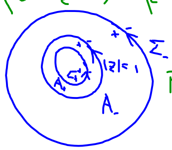
$$\text{Set } N^n(z) := \begin{cases} M^n(z)\begin{pmatrix} z^{-n} & 0 \\ 0 & z^n\end{pmatrix}, & |z|>1 \\ M^n(z), & |z|<1\end{cases}$$

Then, $N^n(z) \to \mathbb{I}$ as $z\to\infty$.
By direct calculation,

$$N^n_+(z) = N^n_-(z)\begin{pmatrix} e^{in\theta} & \phi(\theta)=e^{-V(\theta)} \\ 0 & e^{-in\theta}\end{pmatrix}, \quad z=e^{i\theta}.$$

Now note the factorization:

$$\begin{pmatrix} e^{in\theta} & e^{-V(\theta)} \\ 0 & e^{-in\theta}\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{-in\theta}e^{V(\theta)} & 1\end{pmatrix}\begin{pmatrix} 0 & e^{-V(\theta)} \\ -e^{V(\theta)} & 0\end{pmatrix}\begin{pmatrix} 1 & 0 \\ e^{in\theta}e^{V(\theta)} & 1\end{pmatrix}$$




<u>Set:</u>

$$P^n(z) = \begin{cases} N^n(z)\begin{pmatrix} 1 & 0 \\ z^{-n}e^{\tilde V(z)} & 1\end{pmatrix}, & z\in A_- \\ N^n(z)\begin{pmatrix} 1 & 0 \\ -z^n e^{\tilde V(z)} & 1\end{pmatrix}, & z\in A_+ \\ N^n(z), & \text{elsewhere.}\end{cases}$$

<u>Direct Calculation:</u>

$$|z|=1:\ P^n_+(z) = P^n_-(z)\begin{pmatrix} 0 & e^{-\tilde V(z)} \\ -e^{\tilde V(z)} & 0\end{pmatrix}$$

Sim: $\Sigma_-$. $P^n_+(z) = P^n_-(z)\begin{pmatrix} 1 & 0 \\ z^{-n}e^{\tilde V(z)} & 1\end{pmatrix}$

$$\Sigma_+:\ P^n_+(z) = N^n_+(z) = N^n_-(z) = P^n_-(z)\begin{pmatrix} 1 & 0 \\ z^n e^{\tilde V(z)} & 1\end{pmatrix}$$

Now, get rid of jump on the unit circle:

<u>Model problem</u>. Seek $\dot P(z)$ satisfying: $\dot P(z)\to\mathbb{I}$, $z\to\infty$.

$\dot P$ analytic off <u>$|z|=1$</u>.

$$\dot P_+(z) = \dot P_-(z)\begin{pmatrix} 0 & e^{-\tilde V(z)} \\ -e^{\tilde V(z)} & 0\end{pmatrix}, \quad |z|=1.$$

Solve for $\dot P(z)$: Set $\dot Q = \dot P\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}$, $|z|<1$

$\dot Q = \dot P$, $|z|>1$.

$$\dot Q_+(z) = \dot Q_-(z)\begin{pmatrix} e^{-\tilde V(z)} & 0 \\ 0 & e^{\tilde V(z)}\end{pmatrix}.$$

Seeking $\dot Q$ as diagonal $\Rightarrow$

$$\dot Q_{11+} = \dot Q_{11-}e^{-\tilde V(z)} = \dot Q_{11-}\phi(\theta)$$

take log, solve by Cauchy formula!

How well does $\dot{P}(z)$ approximate $P^n(z)$?

Define $H^n(z) := P^n(\underline{z})\,\dot{P}(z)^{-1}$

Prove $H^n(z)$ is close to $\mathbb{I}$.

$H^n(z)$ is not known. But its related to $M^n(z)$ is.

Moreover, $M^n(z)$ satisfies a RHP

$\Rightarrow H^n(z)$ also satisfies a RHP.

<u>Direct Calc:</u> $H_+^n(z) = H_-^n(z)\,V(z)$, on $\Sigma_+$ and $\Gamma$.

also $V(z)-\mathbb{I}$ is unif. exp smaller as $n\to\infty$.

$H^n(z)\to\mathbb{I}$ as $z\to\infty$.

Translate this RHP into a singular integral equation with Cauchy Kernel.

$\Rightarrow$ for large $n$, solve the RHP by iteration of the int. eqns. Neumann Series, asymptotic for large $n$.

$\Rightarrow$ away from $\Sigma_+$, $H^n(z)-\mathbb{I}$ is unif. exp. small.

Use a "$\bar\partial$" method to analyze non-analytic weights:

<u>Result:</u> let $p\geq 0$ be a fixed integer, let $e^{-V(\theta)} = d(\theta)$ where $V: S^1 \to \mathbb{R}$ is of class $C^{k-1,1}(S^1)$ for some $k\geq 2p+1$. then for some const $K>0$

$$\sup_{|z|\geq 1}\left|\frac{d^p}{dz^p}\left[\pi_n(z)\,z^{-n}e^{-N(z)} - 1\right]\right| \leq K\cdot\frac{\log(n)}{n^{k-2p}}$$

$N(z)$ = negative freq. component of $V(z)$:

$V(\theta) = N(e^{i\theta}) + V_0 + N(e^{-i\theta})$.