(1) Dumitriu & Edelman
construction of a matrix
model for β-ensemble

(2) Ramirez, Rider, Virag
β-Hermite $\longrightarrow$ stochastic Airy operator

	$\beta = 1$ Real	$\beta = 2$ Complex	$\beta = 4$ Quaternion
Hermite	GOE	GUE	GSE
Laguerre	Wishart	Wishart	Wishart
Jacobi	MANOVA	.	.

$X_{n_1 \times p}$ $Y_{n_2 \times p}$

$$X^T X \left(X^T X + Y^T Y \right)^{-1}$$

β-Hermite : $c_\beta^H \, |\Delta(\underline{\lambda})|^\beta \exp\left(-\frac{1}{2}\sum \lambda_i^2\right)$

β-Laguerre $\quad c_\beta^L \, |\Delta(\underline{\lambda})|^\beta \prod_{i=1}^{m} \lambda_i^{a-p} \cdot \exp\left(-\frac{1}{2}\sum_{i=1}^{m} \lambda_i\right)$

$\left(\underset{m\times n}{X^T} \underset{n\times m}{X}\right) \leftarrow \beta=1,2$

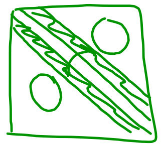
$a = \frac{\beta n}{2} \quad ; \quad p = 1 + \frac{\beta}{2}(m-1)$

Tridiagonalization

$\cdot A \leftarrow$ symmetric

$\downarrow \; HAH^T$

$\cdot T$



GOE: $\quad \underset{n\times n}{A} = \begin{bmatrix} a_n & x_n^T \\ x_n & A_{n-1} \end{bmatrix}$

$\tilde{H}_{n-1} = \begin{bmatrix} 1 & 0 \\ 0 & H_{n-1} \end{bmatrix}$

$\tilde{H}_{n-1} A \tilde{H}_{n-1}^T = \begin{bmatrix} a_n & \cdot \; \text{—} \\ 0 & \vdots \end{bmatrix}$

$H_{n-1} \underline{x}_n = \|\underline{x}_n\|_2 \underline{e}_1$

$$\begin{bmatrix} 1 & 0 \\ 0 & H_{n-1} \end{bmatrix} \begin{bmatrix} a_n & x_n^T \\ x_n & A_{n-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & H_{n-1}^T \end{bmatrix}$$

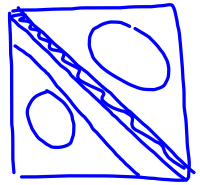
$$= \begin{bmatrix} a_n & x_n^T H_{n-1}^T \\ H_{n-1} x_n & \underline{H_{n-1} A_{n-1} H_{n-1}^T} \end{bmatrix} \overset{d}{=} \begin{bmatrix} a_n & \|x_n\| e_1 \\ \|x_n\| e_1 & A_{n-1} \end{bmatrix}$$

$$H_\beta = \frac{1}{\sqrt{2}} \begin{bmatrix} N(0,2) & \chi_{(n-1)\beta} & 0 & \cdot & 0 \\ \chi_{(n-1)\beta} & N(0,2) & \chi_{(n-2)\beta} & & \\ 0 & \chi_{(n-2)\beta} & N(0,2) & & \\ \vdots & & & \ddots & \\ 0 & & & \chi_\beta & N(0,2) \end{bmatrix}$$

$\underset{(n-1)\times 1}{\underline{x}_n} = \frac{1}{\sqrt{2}} \begin{pmatrix} N(0,1) \\ \vdots \\ N(0,1) \end{pmatrix} \qquad \|x_n\|_2^2 \sim \frac{1}{2} \chi^2_{n-1}$

Laguerre: $(\beta = 1)$.

$$A = BB^T \qquad A = X^T X$$

$B =$ 

$$B = L X^T R^T$$

$$T = \begin{bmatrix} a_n & b_{n-1} & & 0 \\ b_{n-1} & a_{n-1} & b_{n-2} & \\ 0 & \ddots & \ddots & b_1 \\ & & b_1 & a_1 \end{bmatrix} \sim (\underline{a}, \underline{b})$$

$\underline{a} = (a_n, \ldots, a_1)$

$\underline{b} = (b_{n-1}, \ldots, b_1)$

$\underline{\lambda}, \underline{q}$ $\qquad$ $\underline{q} \leftarrow$ first row of Q : $T = Q \Lambda Q^T$

$(\underline{a}, \underline{b}) \longleftrightarrow (\underline{\lambda}, \underline{q})$ a bijection

$$J = \frac{\prod_{i=1}^{n-1} b_i}{\prod_{i=1}^{n} q_i}$$

Paige (Parlett '98)

$$\Delta(\underline{\lambda}) = \prod_{i<j} (\lambda_i - \lambda_j) = \frac{\prod_{i=1}^{n-1} b_i^i}{\prod_{i=1}^{n} q_i}$$

$$dH_\beta = c_\beta \prod_{k=1}^{n-1} b_k^{k\beta-1} \exp\left(-\frac{1}{2}\sum_{i=1}^{n} \lambda_i^2\right) \, da\, db$$

$$= c_\beta \frac{\prod_{i=1}^{n-1} b_i}{\prod_{i=1}^{n} q_i} \prod_{i=1}^{n-1} b_i^{i\beta-1} \exp\left(-\frac{1}{2}\sum_{i=1}^{n}\lambda_i^2\right) \, d\lambda\, dq$$

$$= c_\beta \frac{\prod_{i=1}^{n} b_i^{i\beta}}{\prod_{i=1}^{n} q_i^{\beta}} \cdot \prod_{i=1}^{n} q_i^{\beta-1} \exp\left(-\frac{1}{2}\sum \lambda_i^2\right) \, d\lambda\, dq$$

$$= c_\beta |\Delta(\lambda)|^\beta \prod_{i=1}^{n} q_i^{\beta-1} \exp\left(-\frac{1}{2}\sum^{n} \lambda_i^2\right) \, d\lambda\, dq$$

$\to$ λ and q are independent

$$f_q = c_q^\beta \prod_{i=1}^{n} q_i^{\beta-1} \cdot \quad \text{Dirichlet}(\underbrace{\beta, \ldots, \beta}_{n})$$

$$(q_1^2 \cdots q_n^2) \overset{\propto}{=} \left(\frac{x_1^2}{\sum_{i=1}^{n} x_i^2}, \ldots, \frac{x_n^2}{\sum x_i^2} \right)$$

β-Jacobi:

$$c_\beta \cdot |\Delta(\underline{\lambda})|^\beta \cdot \prod_{i=1}^{n} \lambda_i^{a_1 - \frac{\beta}{2}(n-1)-1} (1-\lambda_i)^{a_2 - \frac{\beta}{2}(n-1)-1}$$

$$a_1 = \frac{\beta}{2} n_1 , \quad a_2 = \frac{\beta}{2} \cdot n_2$$

$$J_\beta = C_\beta C_\beta^T$$

$$C_\beta \sim \begin{pmatrix} c_n & & & & \\ -s_n c'_{n-1} & c_{n-1} & & 0 & \\ & -s_{n-1} c'_{n-2} & \ddots & & \\ 0 & & \ddots & \ddots & \\ & & -s_3 c'_2 & c_2 & \\ & & & -s_2 c'_1 & c_1 \end{pmatrix}$$

$$c_k \sim \sqrt{\mathrm{Beta}\left(\frac{\beta}{2}(a_1+k), \frac{\beta}{2}(a_2+k)\right)}, \quad s_k = \sqrt{1-c_k^2}$$

$$c'_k \sim \sqrt{\mathrm{Beta}\left(\frac{\beta}{2} k, \frac{\beta}{2}(k+a_1+a_2+1)\right)}, \quad s'_k = \sqrt{1-c'^2_k}$$

Edelman + Sutton:

$$\tilde{H}_n^\beta = \frac{\sqrt{2}}{\sqrt{\beta}} n^{1/6}\left(H_n^\beta - \sqrt{2\beta n}\, I\right)$$

Eigenvalue distn of $\tilde{H}_n^\beta$ is "similar" to the eigenvalue distn of

$$H = -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}} b'(x)$$

$$(Hf)\,dx = -f''(x)\,dx + xf(x)\,dx + \frac{2}{\sqrt{\beta}} f(x)\,db(x)$$

b = standard BM.

Thm 1.1 $\lambda_{\beta,1} \geq \lambda_{\beta,2} \geq \cdots$ eigenvalues of H_n^β. Let $\Lambda_0 \leq \Lambda_1 \leq \cdots$ be the spectral points of H. Then the

$$\left\{ \sqrt{\frac{2}{\beta}}\, n^{1/6}\left(\lambda_{\beta,\ell} - \sqrt{2\beta n}\right) \right\}_{\ell=1}^{k}$$

converges in distn. to

$$\{-\Lambda_0, -\Lambda_1, \ldots, -\Lambda_{k-1}\}$$

for all fixed k.

. $H\phi = \lambda\phi$, $\phi(0,\lambda) = 0$. $\phi'(0,\lambda) = 1$

$$\frac{d}{dx}\phi'(x) = \frac{2}{\sqrt{\beta}}\phi(x)\,db_x + (x-\lambda)\phi(x)\,dx$$

Ricatti transform $\phi(x) = e^{\int_0^x p(\cdot)}$

$$p(x) = \frac{\phi'(x)}{\phi(x)} = \frac{d}{dx}(\log\phi(x))$$

$$dp(x) = \frac{2}{\sqrt{\beta}}\,db_x + (x - \lambda - p^2(x))\,dx$$

$$P(\Lambda_0 > \lambda) = P_{+\infty}(\phi(\cdot,\lambda) \text{ never vanishes})$$
$$= P_{+\infty}(p(\cdot,\lambda) \text{ does not explode}).$$

$$\inf\left\{ \langle v, Hv\rangle : \begin{array}{l}\|v\| = 1\\ v(0) = 0\end{array} \right\}$$
$> -C(\omega)$ for some $C(\omega) < \infty$ a.s.

$f_0(0) = 0$ $\qquad \langle \phi, H f_0 \rangle = \Lambda_0 \langle \phi, f_0 \rangle$

$\int f_0^2 = 1$

- $\underline{f_0 \in C^{3/2-}}$:
- f_0 has rapid decay.

$$\left.\begin{aligned} &|f_0(x,\omega)| \\ &|f_0'(x,\omega)| \end{aligned}\right\} \leq c(\omega) \exp(-\gamma x^{2/3})$$

for $0 < \gamma < \frac{2}{3}$

$$E_y\left(e^{\theta\int_0^\infty V(p(x))\,dx}\right)$$

$$\iff \lambda\left(\alpha_0^{x^*} + \theta\right)y_0 < 0$$

$$\frac{1}{Z}\prod_{i<j}|\lambda_i-\lambda_j|^\beta \cdot e^{-\sum V(\lambda_i)}$$

Tridiagonal matrix model

$$\begin{bmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & b_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ & & & b_{n-1} & a_n \end{bmatrix} \qquad J = \frac{\prod_{i=1}^{n-1} b_i}{\prod_{i=1}^{n} q_i}$$

$(a,b) \sim$ pdf χ

$\Rightarrow (\lambda, q)$ has pdf $\chi(a,b)\cdot J \; d\lambda \, dq$

$$\chi(a,b) = \prod_{i=1}^{n-1} b_i^{\beta i - 1} \cdot e^{-Tr(V(A))}$$

$$\Rightarrow \chi(\lambda, q) = \prod |\lambda_i - \lambda_j|^\beta \, e^{-\sum V(\lambda_i)} \cdot f(\underline{q})$$

Eg $V(x) = x^2$ $\quad Tr(V(A)) = \sum a_i^2 + 2\sum b_i^2$

$V(x) = x^4$ $\quad Tr(A^4) = \sum a_i^4 + 2\sum b_i^4$

$$+ 4\sum a_i^2 b_i^2 + 4\sum a_i^2 b_{i-1}^2 + 4\sum a_i a_{i+1} b_i^2 + 4\sum b_i^2 b_{i-1}^2$$