

Identification of Green's functions by cross-correlation of noisy signals

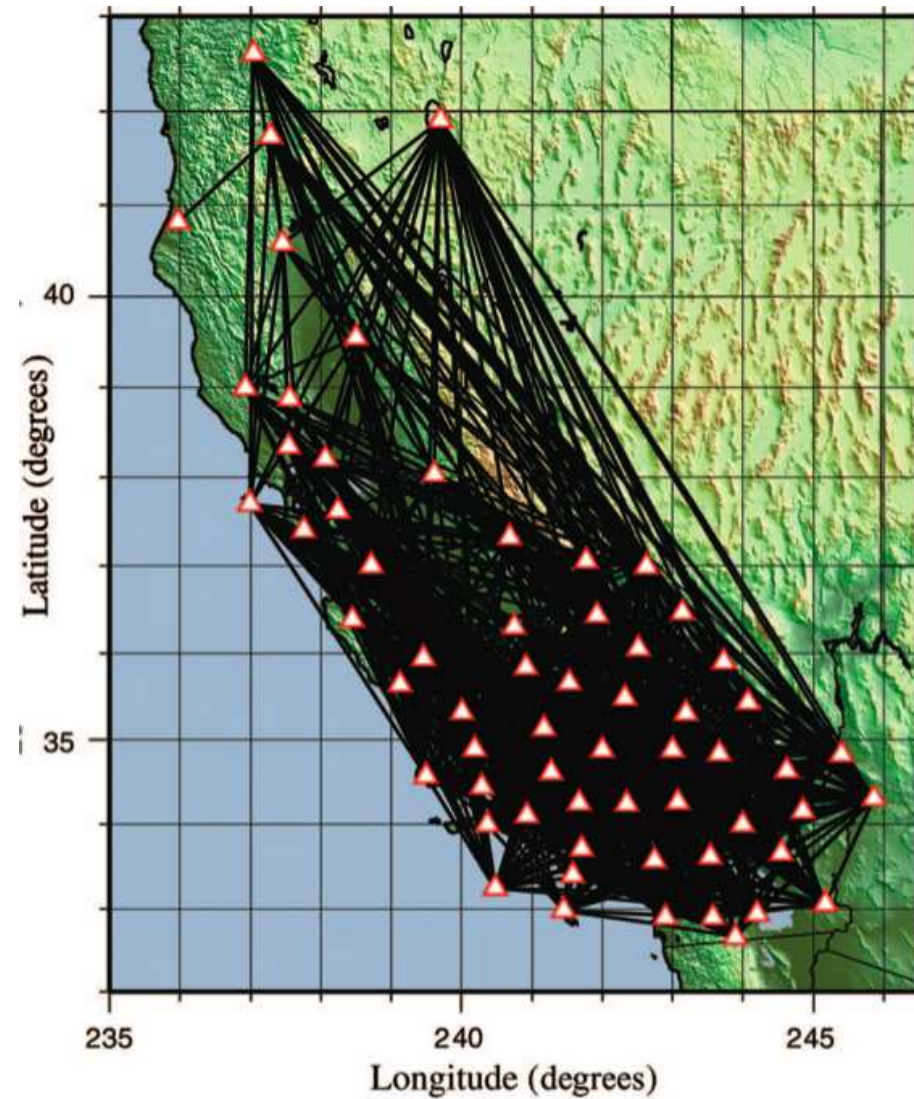
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In a medium in which unknown sources emit noisy signals:

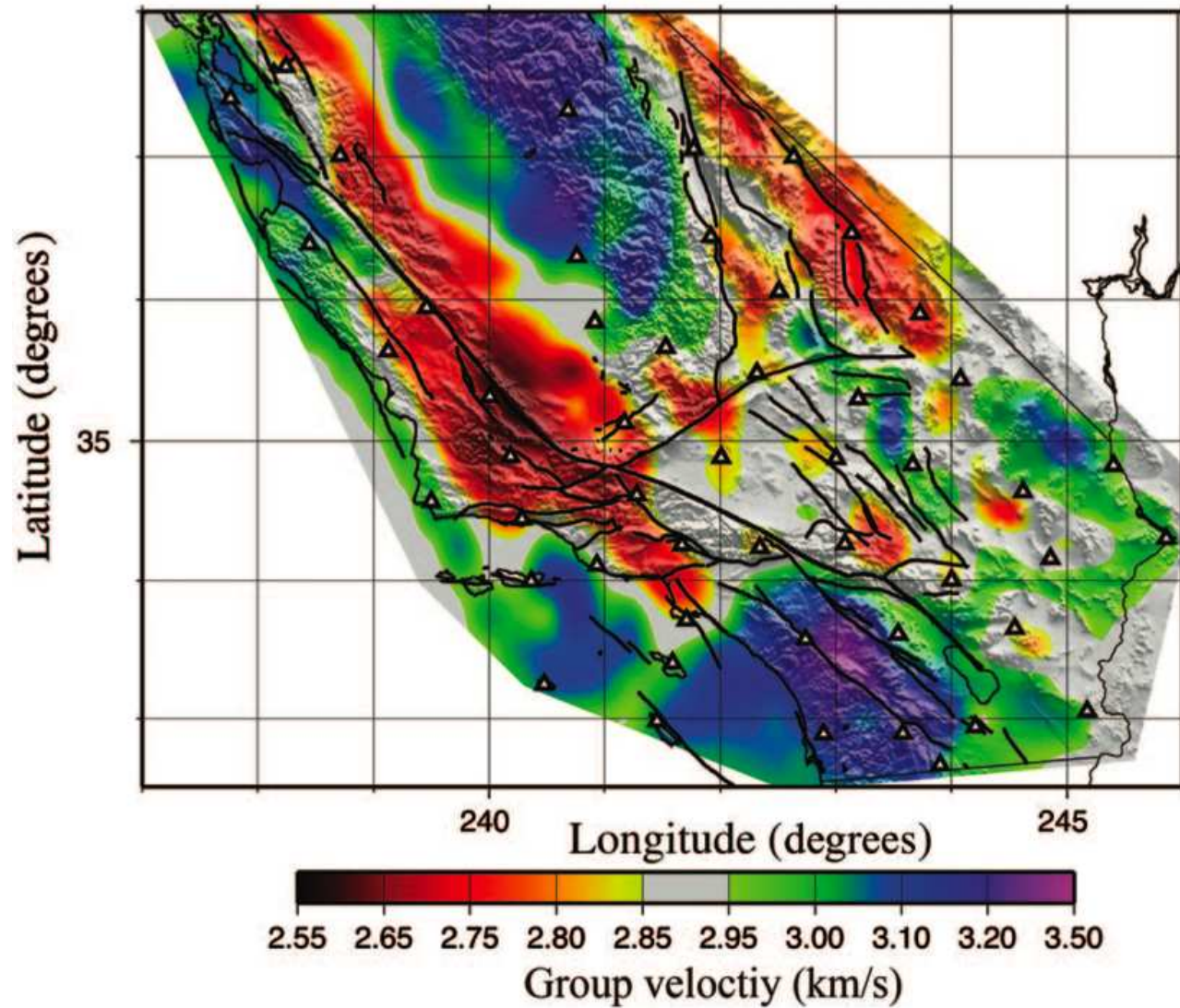
- record (in time) the waves $u(t, x)$ and $u(t, y)$ at two points x and y
- cross-correlate the signals: $C(\tau, x, y) = \frac{1}{T} \int_0^T u(t, x)u(t + \tau, y)dt$
- $C(\tau, x, y)$ is related to the Green's function from x to y !
- the (singular part of the) Green's function from x to y gives the travel time from x to y .

Travel time estimations between pairs of observation points



Surface waves ($d = 2$) [from Larose et al, Geophysics 71, 2006, SI11-SI21]

Background velocity estimation from travel time estimations



[from Larose et al, Geophysics 71, 2006, SI11-SI21]

Wave equation with sources

We consider the solution u of the wave equation with attenuation:

$$\left(\frac{1}{T_a} + \partial_t\right)^2 u - \Delta u = n(t, x) \text{ in } \Omega, \quad u|_{\partial\Omega} = 0$$

- Ω is a bounded open set $\Omega \subset \mathbb{R}^d$.
- the operator Δ is defined by

$$\Delta = \nabla \cdot (c^2(x) \nabla),$$

where c is smooth.

- T_a is the attenuation time.
- $n(t, x)$: noisy sources.

It is a zero-mean stationary (in time) Gaussian process with autocorrelation function

$$\langle n(s, x_s) n(t, y_s) \rangle = \delta(t - s) \theta\left(\frac{x_s + y_s}{2}, x_s - y_s\right)$$

$\langle \cdot \rangle$ is the statistical average over the distributions of the sources.

The function $x_s \mapsto \theta(x_s, 0)$ models the spatial distribution of the sources.

The function $z_s \mapsto \theta(x_s, z_s)$ is the local spatial autocorrelation function.

Example: $\theta(x, y) = \delta(y)$ means "sources everywhere, delta-correlated in space".

The Green's function

- The Green's function is the fundamental solution of

$$\partial_t^2 G - \Delta G = \delta(t)\delta(x - y),$$

starting from $G(0, x, y) = \partial_t G(0, x, y) = 0$.

- It can be written in terms of the eigenvalues ω_n^2 and orthonormal eigenfunctions ϕ_n of $-\Delta$, namely,

$$-\Delta\phi_n = \omega_n^2\phi_n \text{ in } \Omega, \quad \phi_n = 0 \text{ on } \partial\Omega.$$

- It is the distribution

$$G(t, x, y) = \begin{cases} \sum_{n=1}^{\infty} \frac{\sin(\omega_n t)}{\omega_n} \phi_n(x)\phi_n(y) & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$$

which, for $t > 0$, is the kernel of the operator

$$\frac{\sin \sqrt{-\Delta} t}{\sqrt{-\Delta}}.$$

- If $c(x) \equiv c_0$ and $\Omega = \mathbb{R}^d$, then

$$G(t, x, y) = \frac{1}{4\pi c_0^2 |x - y|} \delta\left(\frac{|x - y|}{c_0} - t\right)$$

Note that $\tau_{x,y} = \frac{|x-y|}{c_0}$ is the travel time from x to y .

$$\hat{G}(\omega, x, y) = \frac{1}{4\pi c_0^2 |x - y|} e^{i\omega\tau_{x,y}}$$

- If $c(x)$ is smooth, then (the support of) the singular part of $G(t, x, y)$ contains information on the travel time from x to y .

At high frequency ω , WKB expansion:

$$\hat{G}(\omega, x, y) \sim \sum_{j \geq 0} a^{(j)}(x, y) e^{i\omega\tau_{x,y}^{(j)}}$$

where the sum is over rays $(\phi(t))_{t \in [0,1]}$ connecting x and y that make the action $\tau(\phi) = \int_0^1 \frac{|\phi'(t)| dt}{c(\phi(t))}$ stationary.

In particular, the travel time is

$$\tau_{x,y}^{(0)} = \inf \left\{ \int_0^1 \frac{|\phi'(t)| dt}{c(\phi(t))}, \phi \in \mathcal{C}^2([0, 1], \Omega), \phi(0) = x, \phi(1) = y \right\},$$

We are interested in travel time estimation, therefore we are interested in the high-frequency part of the Green's function.

The (first) key result

The observed signal is

$$u(t, x) = \int_{\Omega} \int_{-\infty}^t n(s, x_s) G(t - s, x, x_s) e^{-\frac{1}{T_a}(t-s)} ds dx_s.$$

Consider the autocorrelation function

$$C_T(\tau, x, y) = \frac{1}{T} \int_0^T u(t, x) u(t + \tau, y) dt$$

- Proposition: If $\theta(x, y) = \delta(y)$ (i.e. sources everywhere, delta-correlated in space), then the average of C_T is an even function in τ that does not depend on T . Its τ -derivative is given by

$$\partial_{\tau} \langle C_T(\tau, x, y) \rangle = -\frac{T_a}{4} \operatorname{sgn}(\tau) G(|\tau|, x, y) e^{-\frac{|\tau|}{T_a}}.$$

An elementary proof of the key result

$$\begin{aligned}
 \langle C_T(\tau, x, y) \rangle &= \frac{1}{T} \int_0^T \langle u(t, x)u(t + \tau, y) \rangle dt \\
 &= \frac{1}{T} \int_0^T \int_{-\infty}^t \int_{-\infty}^{t+\tau} \int_{\Omega^2} \langle n(s, x_s)n(s', x'_s) \rangle \\
 &\quad \times G(t - s, x, x_s)G(t + \tau - s', y, x'_s) e^{-\frac{t-s}{T_a} - \frac{t+\tau-s'}{T_a}} dx_s dx'_s ds' ds dt.
 \end{aligned}$$

The process n is delta-correlated in time and space, so that

$$\langle C_T(\tau, x, y) \rangle = \frac{1}{T} \int_0^T \int_{-\infty}^t \int_{\Omega} G(t - s, x, x_s)G(t + \tau - s, y, x_s) e^{-2\frac{t-s}{T_a}} dx_s ds dt e^{-\frac{\tau}{T_a}}.$$

We make the change of variable $s \mapsto u = t - s$:

$$\langle C_T(\tau, x, y) \rangle = \frac{1}{T} \int_0^T \int_0^\infty \int_{\Omega} G(u, x, x_s)G(u + \tau, y, x_s) e^{-2\frac{u}{T_a}} dx_s du dt e^{-\frac{\tau}{T_a}}.$$

This shows that $\langle C_T \rangle$ does not depend on T :

$$\langle C_T(\tau, x, y) \rangle = \int_0^\infty \int_{\Omega} G(u, x, x_s)G(u + \tau, y, x_s) e^{-2\frac{u}{T_a}} dx_s du e^{-\frac{\tau}{T_a}}.$$

We next substitute the expansion of the Green's function in terms of the eigenvalues and eigenfunctions of $-\Delta$:

$$\begin{aligned} \langle C_T(\tau, x, y) \rangle &= \sum_{n, n'=1}^{\infty} \int_0^{\infty} \frac{\sin(\omega_n u)}{\omega_n} \frac{\sin(\omega'_n(u + \tau))}{\omega'_n} e^{-2\frac{u}{T_a}} du \\ &\quad \times \int_{\Omega} \phi_n(x) \phi_n(x_s) \phi_{n'}(x_s) \phi_{n'}(y) dx_s e^{-\frac{\tau}{T_a}}. \end{aligned}$$

From the orthonormality property of the eigenfunctions

$$\int_{\Omega} \phi_n(x_s) \phi_{n'}(x_s) dx_s = \delta_{nn'}$$

and a direct computation

$$\int_0^{\infty} \frac{\sin(\omega_n u)}{\omega_n} \frac{\sin(\omega_n(u + \tau))}{\omega_n} e^{-2\frac{u}{T_a}} du = \frac{T_a^2}{4\omega_n} \frac{\omega_n T_a \cos(\omega_n \tau) + T_a \sin(\omega_n \tau)}{1 + \omega_n^2 T_a^2}.$$

we obtain

$$\langle C_T(\tau, x, y) \rangle = \sum_{n=1}^{\infty} \frac{T_a^2}{4\omega_n} \frac{\omega_n T_a \cos(\omega_n \tau) + \sin(\omega_n \tau)}{1 + \omega_n^2 T_a^2} \phi_n(x) \phi_n(y) e^{-\frac{\tau}{T_a}}$$

Taking the derivative with respect to τ gives

$$\partial_{\tau} \langle C_T(\tau, x, y) \rangle = -\frac{T_a}{4} \sum_{n=1}^{\infty} \frac{\sin(\omega_n \tau)}{\omega_n} \phi_n(x) \phi_n(y) e^{-\frac{\tau}{T_a}}.$$

$\langle C_T(\tau, x, y) \rangle$ is the kernel of the operator

$$\begin{aligned}
\langle C_T(\tau) \rangle &= \int_0^\infty e^{-\frac{1}{T_a}(\tau+2u)} \frac{\sin \sqrt{-\Delta}u}{\sqrt{-\Delta}} \frac{\sin \sqrt{-\Delta}(u+\tau)}{\sqrt{-\Delta}} \\
&= \int_0^\infty e^{-\frac{1}{T_a}(\tau+2u)} \frac{e^{i\sqrt{-\Delta}(2u+\tau)} - e^{i\sqrt{-\Delta}\tau} - e^{-i\sqrt{-\Delta}\tau} + e^{-i\sqrt{-\Delta}(2u+\tau)}}{-4\Delta} \\
&= \frac{T_a^2}{4\sqrt{-\Delta}} \frac{\sqrt{-\Delta}T_a \cos(\sqrt{-\Delta}\tau) + \sin(\sqrt{-\Delta}\tau)}{1 + (-\Delta)T_a^2} e^{-\frac{\tau}{T_a}}
\end{aligned}$$

$\partial_\tau \langle C_T(\tau, x, y) \rangle$ is the kernel of the operator

$$e^{-\frac{\tau}{T_a}} \frac{\sin \sqrt{-\Delta}\tau}{\sqrt{-\Delta}}$$

The (second) key result

- If $\theta(x, y) = \delta(y)$, then, for any T

$$\langle \partial_\tau C_T(\tau, x, y) \rangle = -\frac{T_a}{4} \text{sgn}(\tau) G(|\tau|, x, y) e^{-\frac{|\tau|}{T_a}}.$$

- Statistical stability

$$\langle \partial_\tau C_T(\tau, x, y)^2 \rangle - \langle \partial_\tau C_T(\tau, x, y) \rangle^2 = O\left(\frac{T_a}{T}\right)$$

Proof: $\langle \partial_\tau C_T(\tau, x, y)^2 \rangle$ can be written as a multiple integral involving forth-order moments of $n(t, x)$. Use the Gaussian property to write the forth-order moment as products of second-order moments:

$$\langle n_1 n_2 n_3 n_4 \rangle = \langle n_1 n_2 \rangle \langle n_3 n_4 \rangle + \langle n_1 n_3 \rangle \langle n_2 n_4 \rangle + \langle n_1 n_4 \rangle \langle n_2 n_3 \rangle$$

Consequence:

$$\mathbb{P}\left(\partial_\tau C_T(\tau, x, y) - \langle \partial_\tau C_T(\tau, x, y) \rangle \geq \delta\right) \leq \frac{KT_a}{\delta^2 T}$$

If

$$T \gg T_a$$

then one realization of $\partial_\tau C_T(\tau, x, y)$ is close to the statistical average $\langle \partial_\tau C_T(\tau, x, y) \rangle$.

What if sources are not everywhere ?

Assume: the sources are neither stationary in space, nor delta-correlated ($\theta(x, y) \neq \delta(y)$).

The previous result is still true if the cavity Ω is classically ergodic, up to a smoothing operator K_θ and up to a remainder R that is small for high frequencies.

Proposition If

- $c \in W^{4,\infty}(\Omega)$,
- $\hat{\theta}(x, \xi) := \int \theta(x, y) e^{-i\xi \cdot y} dy$ is smooth, bounded, and integrable,
- $T_a < 2T_{\text{ergo}}$,

then $\langle \partial_\tau C_T(\tau, x, y) \rangle$ is the kernel of the operator

$$e^{-\frac{\tau}{T_a}} K_\theta \frac{\sin \sqrt{-\Delta} \tau}{\sqrt{-\Delta}} + R_\theta(\tau)$$

$$K_\theta \psi(x) = \int \psi(x - z) \frac{1}{c(x - \frac{z}{2})^d} k_\theta \left(\frac{z}{c(x - \frac{z}{2})} \right) dz, \quad k_\theta(y) = \frac{\int_\Omega \theta(z', c(z')y) dz'}{\int_\Omega c(z')^{-d} dz'}$$

The remainder $R_\theta(\tau)$ is small: there exists a constant $C_R(c, \theta)$ such that

$$\|R_\theta(\tau)\psi\|_{H^1(\Omega)} \leq C_R e^{-\frac{\tau}{T_a}} \left[r_{\text{erg}}(T_a) + \frac{1}{1 - \frac{T_a}{2T_{\text{ergo}}}} \frac{\|\psi\|_{H^{-1}(\Omega)}}{\|\psi\|_{L^2(\Omega)}} \right] \|\psi\|_{L^2(\Omega)},$$

where $r_{\text{erg}}(t)$ goes to 0 as $t \rightarrow \infty$.

- Three ingredients:
 - approximate the wave dynamics by classical dynamics (Egorov's theorem): the high-frequency components propagate along the classical rays.
 - Ω is an ergodic cavity: the classical dynamics along the rays is ergodic (starting from any point, any direction, the ray visits all of the phase space).
 - the noisy sources $n(t, x)$ is a Gaussian process: the forth-order moments are products of second order moments.
- Problem of time scales:
 - Egorov's theorem breaks down after time T_{ego} ,
 - ergodicity needs some time T_{erg} (and no uniform estimate !),
 - self-averaging property needs an integration time long enough ($T \gg T_a$).

First ingredient: ergodicity of the classical flow

Classical Hamiltonian flow gives the propagation of rays

$$(x, \xi) \mapsto e^{th}(x, \xi) = (x_t(x, \xi), \xi_t(x, \xi)),$$

with the Hamiltonian $h(x, \xi) = \sqrt{(c^2(x)\xi, \xi)} = c(x)|\xi|$:

$$\begin{aligned} \frac{dx_t}{dt} &= c(x_t) \frac{\xi_t}{|\xi_t|}, & x_0(x, \xi) &= x, \\ \frac{d\xi_t}{dt} &= -\nabla c(x_t)|\xi_t|, & \xi_0(x, \xi) &= \xi. \end{aligned}$$

+ rays are reflected on the boundary $\partial\Omega$ according to the Snell's law.

The Hamiltonian flow e^{th} is said to be (classically) ergodic if for any $f \in L^\infty(S^*(\Omega))$ and for $m = (x, \xi)$ in a subset of full measure of $S^*(\Omega)$,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(e^{sh}m) ds = \bar{f} := \frac{1}{\mu(S^*(\Omega))} \int_{S^*(\Omega)} f(m) d\mu(m).$$

where $S^*(\Omega)$ is the cotangent spherical bundle (energy surface)

$$S^*(\Omega) = \{(x, \xi) \in T^*\Omega, c(x)|\xi| = 1\}$$

Second ingredient: Egorov's theorem

The function $\theta(x, y)$ defines the covariance operator $\Theta : L^2(\Omega) \rightarrow L^2(\Omega)$

$$\Theta\psi(x) = \int \theta\left(\frac{x+y}{2}, x-y\right)\psi(y)dy,$$

which is a zero-order pseudodifferential operator with symbol $\hat{\theta}(x, \xi)$

$$\Theta = \text{Op}(\hat{\theta}(x, \xi)).$$

Here we have used the Weyl quantization Op defined by

$$\text{Op}(\hat{\theta}(x, \xi))\psi(x) = \frac{1}{(2\pi)^d} \int \hat{\theta}\left(\frac{x+y}{2}, \xi\right) e^{i\xi \cdot (x-y)} \psi(y) dy d\xi.$$

The semiclassical Egorov theorem states that the operator

$$e^{it\sqrt{-\Delta}}\Theta e^{-it\sqrt{-\Delta}} - \text{Op}\left[\hat{\theta}(e^{th}(x, \xi))\right]$$

is continuous from $H^{-1}(\Omega)$ to $L^2(\Omega)$ with a norm which grows in t with an exponential rate $1/T_{ego}$.

$$\begin{aligned}
\langle \partial C_T(\tau) \rangle &= K_1 + K_2, \\
K_1 &= -\frac{4}{T_a} \int_0^\infty e^{-\frac{1}{T_a}(\tau+2s)} \frac{\sin(\sqrt{-\Delta}s)}{\sqrt{-\Delta}} \Theta \cos(\sqrt{-\Delta}(s+\tau)) ds, \\
K_2 &= -\frac{4}{T_a^2} \int_0^\infty e^{-\frac{1}{T_a}(\tau+2s)} \frac{\sin(\sqrt{-\Delta}s)}{\sqrt{-\Delta}} \Theta \frac{\sin(\sqrt{-\Delta}(s+\tau))}{\sqrt{-\Delta}} ds.
\end{aligned}$$

We write K_1 in the form

$$K_1 = e^{-\frac{\tau}{T_a}} K_\theta \frac{\sin \sqrt{-\Delta} \tau}{\sqrt{-\Delta}} + \frac{e^{-\frac{\tau}{T_a}}}{\sqrt{-\Delta}} (R_1 + R_2 + R_3),$$

$$\begin{aligned}
R_1 &= \frac{i}{T_a} \int_0^\infty \left[\text{Op}(\hat{\theta}(e^{sh}(x, \xi))) - K_\theta \right] e^{-\frac{2s}{T_a}} ds e^{-i\tau\sqrt{-\Delta}} \\
&\quad - \frac{i}{T_a} \int_0^\infty \left[\text{Op}(\hat{\theta}(e^{-sh}(x, \xi))) - K_\theta \right] e^{-\frac{2s}{T_a}} ds e^{i\tau\sqrt{-\Delta}}, \\
R_2 &= \frac{i}{T_a} \int_0^\infty \left[e^{-is\sqrt{-\Delta}} \Theta e^{is\sqrt{-\Delta}} - \text{Op}(\hat{\theta}(e^{-sh}(x, \xi))) \right] e^{-\frac{2s}{T_a}} ds e^{-i\tau\sqrt{-\Delta}} \\
&\quad - \frac{i}{T_a} \int_0^\infty \left[e^{is\sqrt{-\Delta}} \Theta e^{-is\sqrt{-\Delta}} - \text{Op}(\hat{\theta}(e^{sh}(x, \xi))) \right] e^{-\frac{2s}{T_a}} ds e^{i\tau\sqrt{-\Delta}}, \\
R_3 &= \frac{i}{T_a} \int_0^\infty e^{-is\sqrt{-\Delta}} \Theta e^{-is\sqrt{-\Delta}} e^{-\frac{2s}{T_a}} ds e^{-i\tau\sqrt{-\Delta}} \\
&\quad - \frac{i}{T_a} \int_0^\infty e^{is\sqrt{-\Delta}} \Theta e^{is\sqrt{-\Delta}} e^{-\frac{2s}{T_a}} ds e^{i\tau\sqrt{-\Delta}}.
\end{aligned}$$

$$\begin{aligned}
\langle \partial C_T(\tau) \rangle &= K_1 + K_2, \\
K_1 &= -\frac{4}{T_a} \int_0^\infty e^{-\frac{1}{T_a}(\tau+2s)} \frac{\sin(\sqrt{-\Delta}s)}{\sqrt{-\Delta}} \Theta \cos(\sqrt{-\Delta}(s+\tau)) ds, \\
K_2 &= -\frac{4}{T_a^2} \int_0^\infty e^{-\frac{1}{T_a}(\tau+2s)} \frac{\sin(\sqrt{-\Delta}s)}{\sqrt{-\Delta}} \Theta \frac{\sin(\sqrt{-\Delta}(s+\tau))}{\sqrt{-\Delta}} ds.
\end{aligned}$$

We write K_1 in the form

$$K_1 = e^{-\frac{\tau}{T_a}} K_\theta \frac{\sin \sqrt{-\Delta} \tau}{\sqrt{-\Delta}} + \frac{e^{-\frac{\tau}{T_a}}}{\sqrt{-\Delta}} (R_1 + R_2 + R_3),$$

$$\begin{aligned}
R_1 &= \frac{i}{T_a} \int_0^\infty \left[\text{Op}(\hat{\theta}(e^{sh}(x, \xi))) - K_\theta \right] e^{-\frac{2s}{T_a}} ds e^{-i\tau\sqrt{-\Delta}} \quad \text{ergodic theorem for } \hat{\theta} \\
&\quad - \frac{i}{T_a} \int_0^\infty \left[\text{Op}(\hat{\theta}(e^{-sh}(x, \xi))) - K_\theta \right] e^{-\frac{2s}{T_a}} ds e^{i\tau\sqrt{-\Delta}},
\end{aligned}$$

$$\begin{aligned}
R_2 &= \frac{i}{T_a} \int_0^\infty \left[e^{-is\sqrt{-\Delta}} \Theta e^{is\sqrt{-\Delta}} - \text{Op}(\hat{\theta}(e^{-sh}(x, \xi))) \right] e^{-\frac{2s}{T_a}} ds e^{-i\tau\sqrt{-\Delta}} \quad \text{Egorov} \\
&\quad - \frac{i}{T_a} \int_0^\infty \left[e^{is\sqrt{-\Delta}} \Theta e^{-is\sqrt{-\Delta}} - \text{Op}(\hat{\theta}(e^{sh}(x, \xi))) \right] e^{-\frac{2s}{T_a}} ds e^{i\tau\sqrt{-\Delta}},
\end{aligned}$$

$$\begin{aligned}
R_3 &= \frac{i}{T_a} \int_0^\infty e^{-is\sqrt{-\Delta}} \Theta e^{-is\sqrt{-\Delta}} e^{-\frac{2s}{T_a}} ds e^{-i\tau\sqrt{-\Delta}} \quad \text{off-resonant} \\
&\quad - \frac{i}{T_a} \int_0^\infty e^{is\sqrt{-\Delta}} \Theta e^{is\sqrt{-\Delta}} e^{-\frac{2s}{T_a}} ds e^{i\tau\sqrt{-\Delta}}.
\end{aligned}$$

Summary

If

$$T_{erg} \ll T_a < 2T_{ego} \text{ and } T_a \ll T$$

where

- T_{erg} is the convergence time for the ergodic theorem applied to $\hat{\theta}(e^{th}(x, \xi))$:

$$\left\| \frac{1}{t} \int_0^t \hat{\theta}(e^{sh}(x, \xi)) ds - \bar{\theta} \right\| \leq r\left(\frac{t}{T_{erg}}\right)$$

- T_a is the attenuation time of the medium,

- T_{ego} is the Egorov time for $\hat{\theta}$:

$$\left\| e^{it\sqrt{-\Delta}} \Theta e^{-it\sqrt{-\Delta}} - \text{Op} \left[\hat{\theta}(e^{th}(x, \xi)) \right] \right\|_{H^{-1} \rightarrow L^2} \leq C_{ego} e^{t/T_{ego}},$$

- T is the integration time,

then everything works !

If we restrict the set of test functions to the high-frequency set

$$B_\delta = \left\{ \psi \in L^2(\Omega), \|\psi\|_{L^2(\Omega)} = 1, \|\psi\|_{H^{-1}(\Omega)} \leq \delta \right\}, \text{ where } \delta \ll 1$$

then

$$\partial_\tau C_T(\tau) \sim e^{-\frac{\tau}{T_a}} K_\theta \frac{\sin \sqrt{-\Delta} \tau}{\sqrt{-\Delta}}$$

Surface waves

- The full theory for surface waves has not been written, not the case of sources located at the boundary.
- In fact, it is possible to reconstruct 3D maps of background velocity with surface waves !

Fact: low-frequency surface waves penetrate deeper in the earth crust than high-frequency surface waves.

Method:

- filter the Green's function in frequency bands
- for each frequency band, compute the travel times
- each frequency band corresponds to a particular penetration depth of the surface waves.

- Segmentation problem: assume that we have one-year signals.
 - solution 1: cross-correlate one-year signals,
 - solution 2: cross-correlate one-month signals and average over 12 months.Solution 2 definitely better in case of seasonal sources, and better in general.
What is the optimum ?