A FM Aproach to Sensing and Imaging

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Introduction - nonlinear problems

- Don't know how to solve general nonlinear problems

- Solvable Problems
  - (0) Quadratic formula
  - (1) Polynomial root finding
  - (2) Eigenvalue problems
  - (3) $\cdots$

- Many applications rely on these fundamental procedures

Introduction - Euler and Prony

- Euler's and Proney's methods for the forward and inverse problems of constant coefficient ODEs

(i) Euler: Given ODE, find solution $u(t)$ as sum of exponentials

(ii) Prony: Given $u(t)$, find its ODE and use Euler's method
to determine frequencies $\{\,\omega_j\,\}$ and amplitudes $\{\,c_j\,\}$ in

$$u(t) = \textstyle\sum_{j=1}^{n} \; c_j \; e^{i\omega_j t}$$

Introduction - Euler and Prony

- Euler used a nonlinear procedure to solve a linear ODE
- Prony's method: Earliest inverse problem solver for ODEs

- How are these methods faring today, not quite well

  (i) Euler's method: Hardly used for numerical ODEs and PDEs (as compared with the other Euler's method which is widely used for numerical ODEs and PDEs)

  (ii) Prony's method: Not used for imaging or inverse scattering

1. *In this talk, we'll show how to use Prony's method for sensing and imaging, signal processing and compression.*

2. *Prony's original method is not quite stable, but has more stable variants, such as HSVD*

Introduction - Prony's method

- Prony actually fared much better than Euler

- It has been refined and widely used in spectroscopy

- Variants of Prony: Szego's algorithm, AAK theory, Pade Transform, Pisarenko's algorithm, HSVD ...

- HSVD relies on eigenvalue decomposition. It can be *regularized*, and *generalized* to higher dimensions.

Introduction - Prony's method

♢ The problem in 1-D

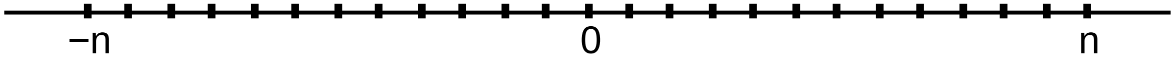
Given $g(x) = \sum_{j=1}^{n} c_j \, e^{ik_j x}$ as a sum of exponentials,
determine the frequencies $\{ \, k_j \, \}$ and coefficients $\{ \, c_j \, \}$

♢ The problem in 2-D

Given $g(x,y) = \sum_{j=1}^{n} c_j \, e^{i(u_j x + v_j y)}$ as a sum of exponentials,
determine the frequencies $\{ \, (u_j, v_j) \, \}$ and coefficients $\{ \, c_j \, \}$

The solution by Prony, 1-D, year 1795

1. Sample g on uniform grid, say, at $2n+1$ integer points



2. Construct a Toeplitz matrix T with $[\,g(-n), g(-n+1), \ldots, g(n)\,]$ showing up on the anti-diagonal

3. Pick a vector $a = (a_0, a_1, \ldots, a_n)$ from the null space of T, and define polynomial of deg n: $P(z) = a_0\ z^n + a_1\ z^{n-1} + \cdots + a_n$

4. The n roots are $r_j = \exp(\omega_j)$. Find the roots and obtain ω_j

The HSVD algorithm, 1-D, year 1983

Problem: Fit f(t) by exponentials, $f(t) = \sum_{j=1}^{n} c_j \, e^{i\omega_j t}$

Algorithm. Sample f uniformly: $f_\ell = f(t_\ell)$, $t_\ell = t_0 + \ell h$

1. Fold $[f_1, f_2, \ldots f_{2m+1}]$ into a Hankel matrix H of size $m+1$
2. SVD on H: $H = U \, \Sigma \, V^*$. Only keep significant columns of U

3. Remove top row of U, and call it U_{top}.
   Remove bottom row of U, and call it U_{bott}

4. Find the eigenvalues of the quotient $U_{\text{bott}}^{+} \, U_{\text{top}}$
5. These eigenvalues are $\lambda_j = e^{ih\omega_j}$. Determine ω_j from λ_j

(The quotient $U_{\textbf{bott}}^{+} \, U_{\textbf{top}}$ is the shift operator $S : f(t) \mapsto f(t+h)$, whereas exponentials are eigen-functions of S)

1. *HSVD has been extended to higher dimensions, year 2007*

2. *This is no small deal*

The following subjects are closely related

1. Prony's method (HSVD)
2. Image point targets in any dimension from (partial) Fourier data
3. Design Gaussian quadratures for bandlimited functions in any dimension, for arbitrary weight and domain shape
4. Remove Gibbs phenomenon in Fourier inversion

In general, Prony's method is capable of extracting features in image from the Fourier or scattering data

Some terms used in this talk

1. Imaging: Reconstruct a function from its Fourier data

2. Features in image: Anything that causes Fourier data to decay slowly - Point targets, interfaces of discontinuities, etc

2.1. Features have easily discernable locations

3. FM analysis on a function: To fit it with sum of exponentials $g(t) = \sum_{j=1}^{n} c_j \, e^{i\omega_j t}$ in 1-D, or $g(u,v) = \sum_{j=1}^{n} c_j \, e^{i(ux_j + vy_j)}$ in 2-D

4. Prony's method: Nonlinear Fourier analysis to do FM

. . . will use the terms "Prony" and "FM" interchangeably

An irony in imaging - AM approach

- Images are very largely defined by features

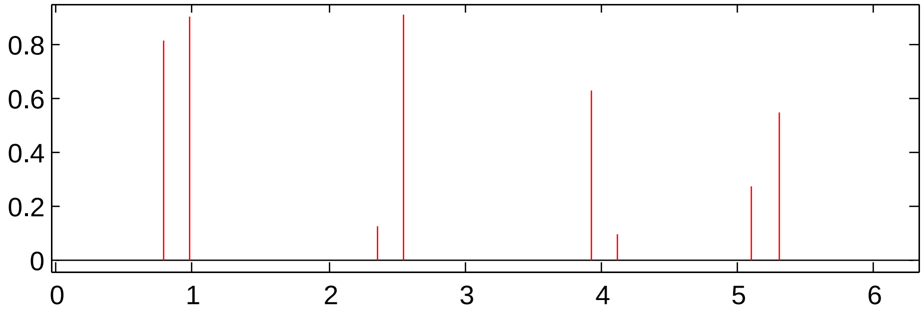
- Imaging should be very largely about locating these features

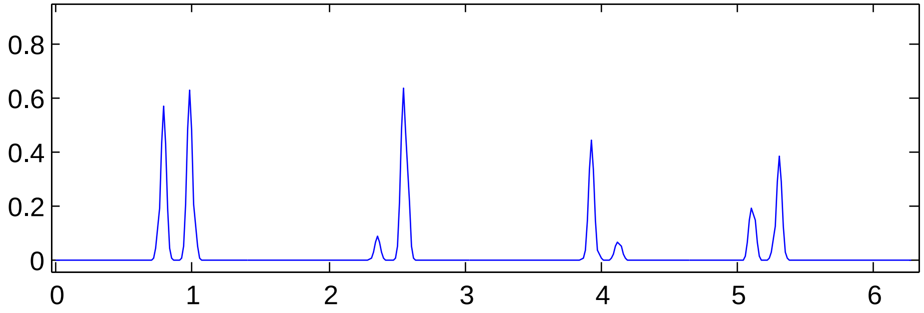
- But no imaging method asks location questions explicitly

- AM approach: Use amplitude of some function to indicate locations of features in image

AM approach to imaging

AM: Function value to indicate locations and strengths of point targets in $[0, 2\pi]$


The locations and strengths of 8 point targets


A function to indicate the locations and strengths

- Typical AM methods: Fourier inversion, Time reversal

FM approaches to imaging

- Many AM methods work pretty well for overwhelming Fourier data

- They work not so well if data, aperture, or bandwidth are sufficient but not overwhelming

- FM: Deal with locations of features directly

- Asking locations questions entails nonlinearity

1. How to ask locations questions
2. What nonlinearity to encounter
3. What are the advantages and drawbacks for FM

Plan of talk

♢ FM to image point targets

♢ Compare FM and AM modalities for imaging

♢ FM analysis for Gaussian quadrature design

♢ FM analysis for Gibbs phenomenon

♢ Conclusion and Discussion

A simple imaging problem

♢ A fundamental problem in imaging : Solve for f given s

$$s(k) = \int_{-\pi}^{\pi} e^{-ikx} f(x)dx, \quad k \in [-k_0, k_0]$$

f is the image, s is the signal (Fourier data)

$k \in [-k_0, k_0]$, k often assumes integer values

♢ In 2-D, $x \hookrightarrow (x, y)$, $k \hookrightarrow (u, v)$

A simple question

- ♢ **Is it a linear problem** : $s(k) = \int_{-\pi}^{\pi} e^{-ikx} f(x) dx$?
- ♢ Yes, Fourier inversion recovers f well for smooth f
- ♢ Yes, and it can be done fast with FFT

A simple question

♢ **Is it a linear problem** : $s(k) = \int_{-\pi}^{\pi} e^{-ikx} f(x) dx$?

♢ No, what if f has jump discontinuities: Fourier inversion could be awful : Gibbs oscillations - still unresolved problem

♢ No, what if f consists of point targets: Fourier inversion is bad

♢ No, what if Fourier data have low frequency part missing : Limited aperture data

If imaging requires a nonlinear procedure,

what then is the type of nonlinearity ?

Point targets, Frequency modulation

♢ When f consists of n point targets

$$f(x) = \sum_{j=1}^{n} c_j \, \delta(x - x_j),$$

then the signal

$$s(k) = \int_{-\pi}^{\pi} e^{-ikx} f(x) dx = \sum_{j=1}^{n} c_j \, e^{-ikx_j}$$

is a superposition of n pure tones $e^{-ikx_j}, \quad j = 1, 2, \ldots, n$

♢ The j-th pure tune e^{-ikx_j} has frequency x_j

- Locations $\{x_j\}$ are encoded in the phase of signal
- Signal $s(k)$ is frequency modulated by locations $\{x_j\}$

Frequency modulation

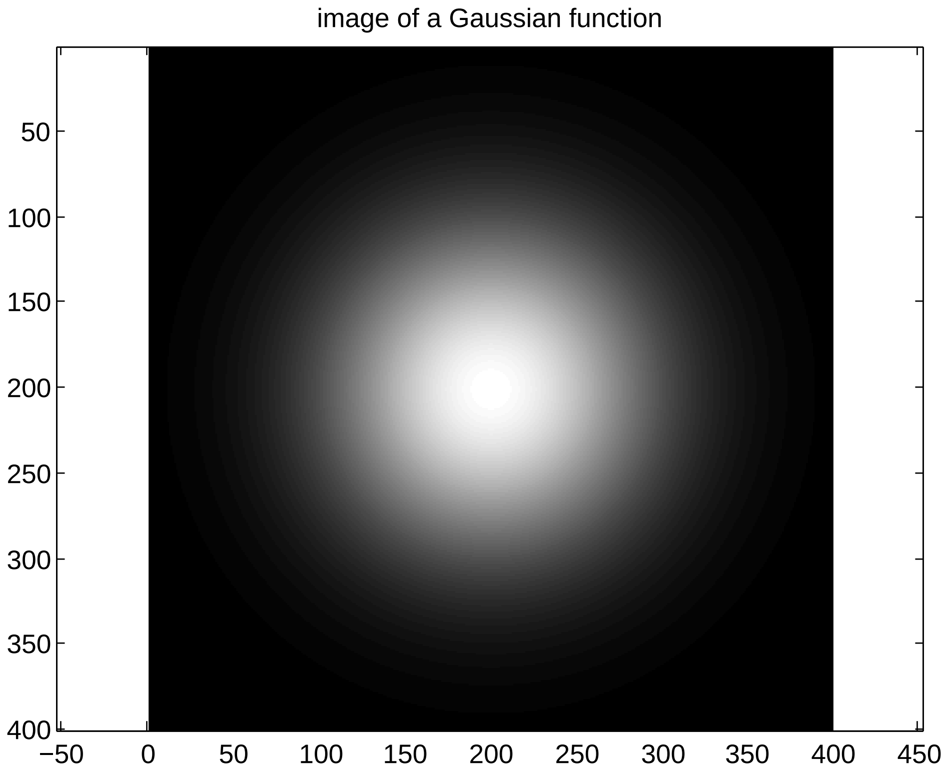
♢ Same story for jump discontinuities : Locations of jumps are nonlinearly encoded in the phase of signal $s(k)$

♢ In general, locations of features in image are nonlinearly encoded in Fourier data by frequency modulation

♢ Only smooth part of image is linearly encoded

Question: What is the generic function most easily recovered from its Fourier data by IFFT?

Image of a Gaussian - a featureless, textureless fuzzball



Ok, it is nonlinear, it is frequency modulation,

but is there a reliable method to handle it ?

Prony’s method

$\diamondsuit$ Prony’s method is tailor made to image point targets in 1 and higher dimensions

$\diamondsuit$ Prony’s method is also useful to locate discontinuities : edges, interfaces separating different materials

FM vs AM in imaging

♢ Imaging point targets $f(x) = \sum_j c_j \, \delta(x - x_j)$ from Fourier data

using AM formulation $s(k) = \int_{-\pi}^{\pi} e^{-ikx} f(x) dx$
or FM formulation $s(k) = \sum_j c_j \, e^{-ikx_j}$

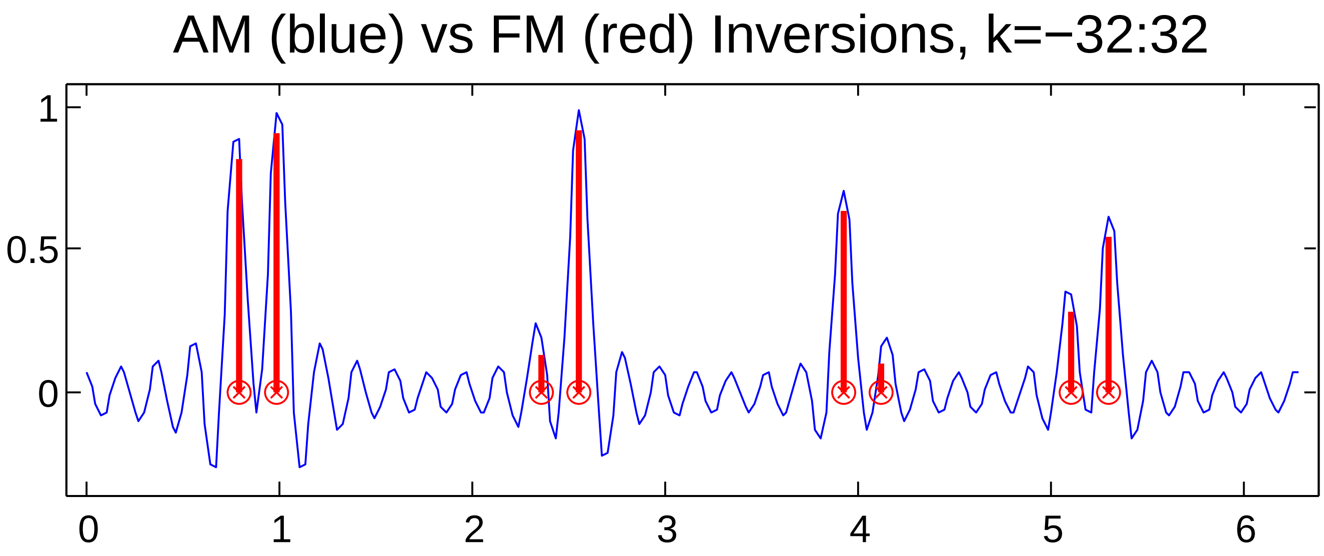
♢ They are equivalent in producing the Fourier data

♢ But not equivalent as forward models for inversion

♢ Both represent signal as superposition of Fourier basis, but

- AM: e^{-ikx} is pre-selected; x is a free variable
- FM: e^{-ikx_j} is to be selected in the process of imaging; x_j is unknown

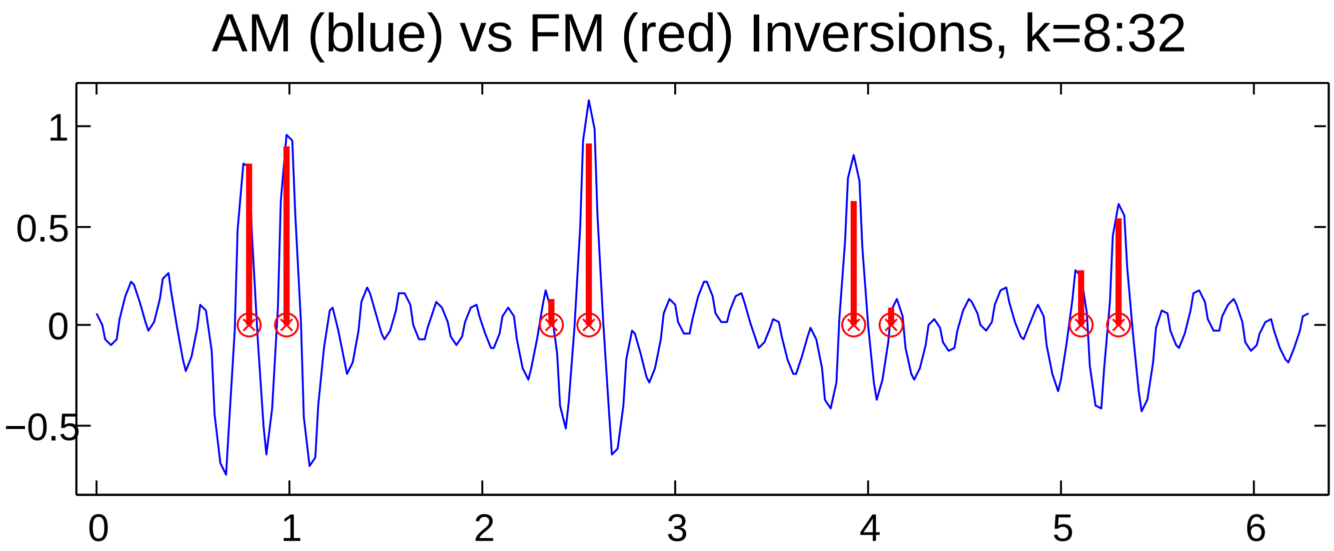
Example 1: Imaging point targets by AM and FM



AM: Point spread function rings a lot

FM: No such concept as point spread function.
Reconstruction is exact.

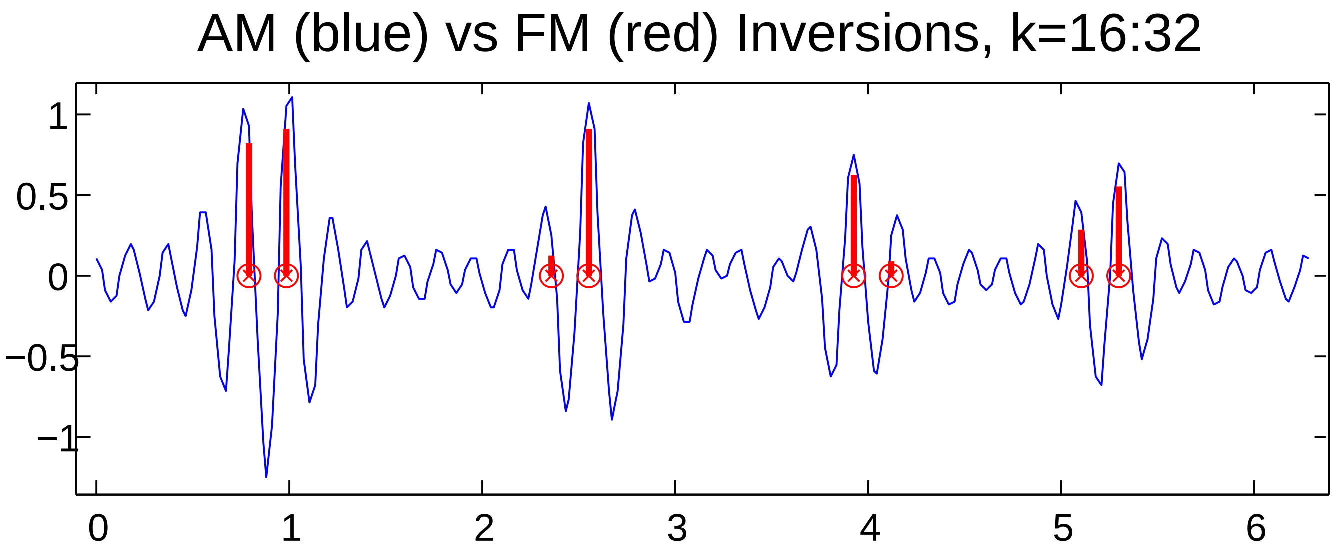
Example 2: Low frequencies are missing



AM: Getting worse

FM: Reconstruction is exact.

Example 3: More low frequencies are missing



AM: Getting worse

FM: Reconstruction is exact.

FM vs AM in imaging

To reconstruct $f(x) = \sum_j c_j \, \delta(x - x_j)$ using

AM: $s(k) = \int_{-\pi}^{\pi} e^{-ikx} f(x) dx$ or FM: $s(k) = \sum_j c_j \, e^{-ikx_j}$

♢ AM: Use amplitude f to represent location x_j and strength c_j

♢ AM: Point spread functions don't behave well.
Reconstruction is riddled with Gibbs oscillations

♢ FM: Point spread functions don't spread at all.
Reconstruction is exact

FM over AM approaches

♢ AM : Linear Fourier analysis
♢ FM : Nonlinear Fourier analysis

♢ AM : Low resolution, Ringing, Point spread function
♢ FM : Highest resolution - exact for point targets

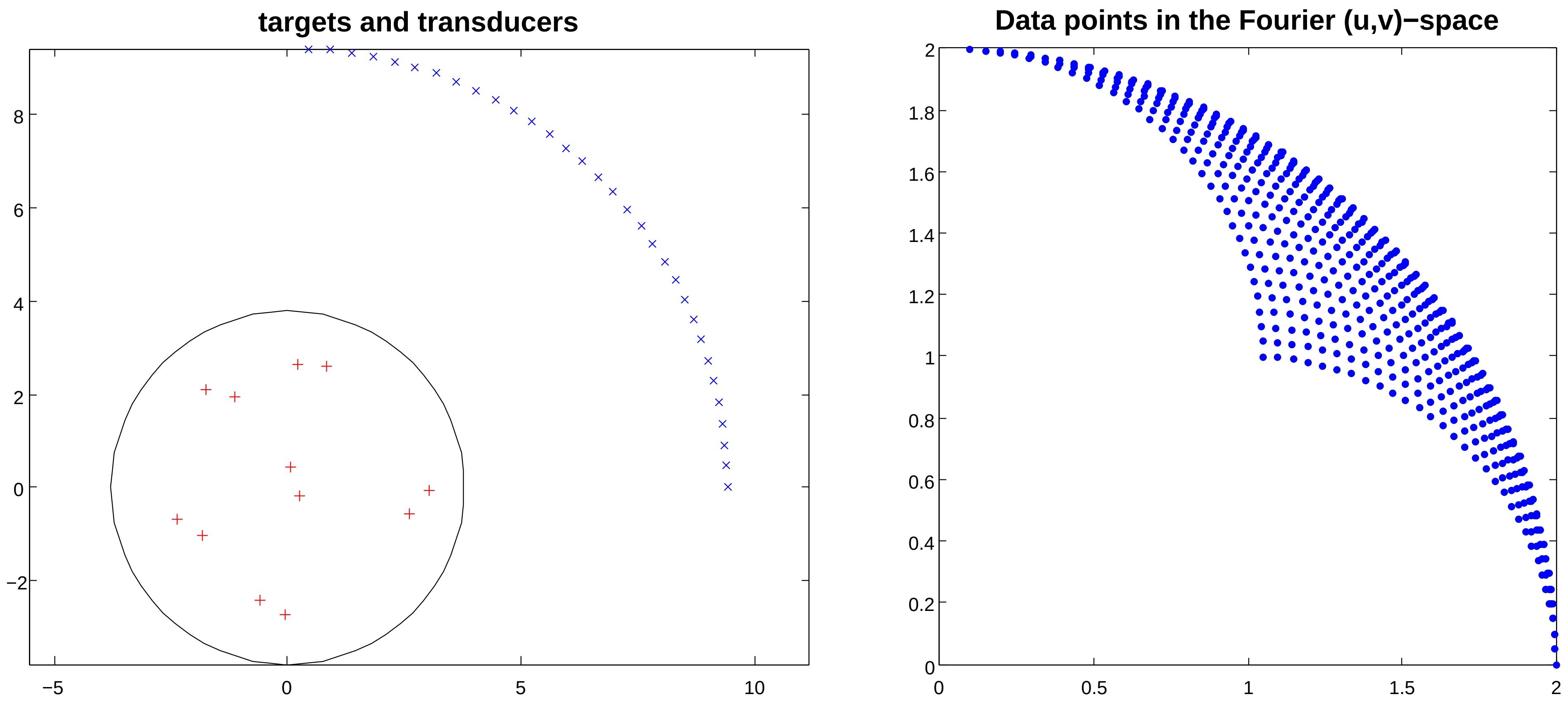
♢ AM : Resolution worsens for limited aperture
♢ FM : Exact reconstruction for limited aperture

♢ AM : Extremely widely used for sensing and imaging
♢ FM : Rarely used for imaging, little is known about it

FM to image point targets in higher dimensions

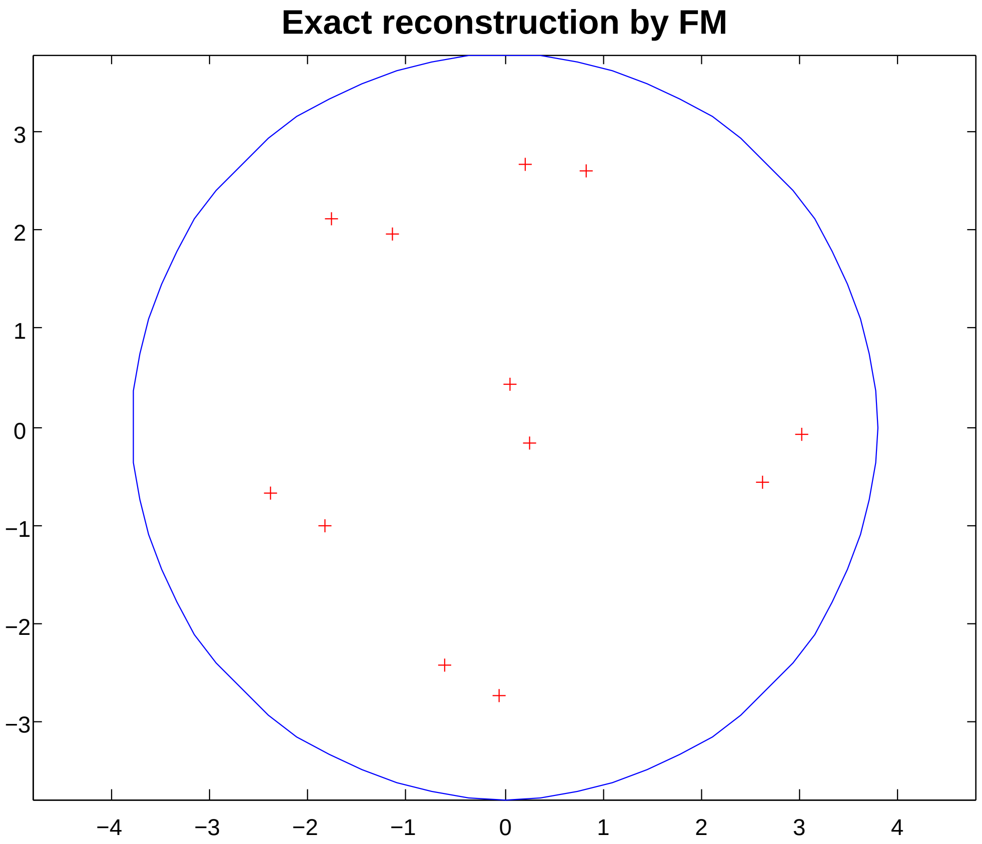
- ♢ Higher dimensional FM analysis on signal s: To fit it by sum of bivariate exponentials $s(u,v) = \sum_{j=1}^{n} c_j \, e^{i(ux_j + vy_j)}$
- ♢ Done with HSVD method extended to 2 and 3-D
- ♢ Exact reconstruction of point targets with partial aperture

FM analysis for a 2-D imaging problem



- To determine locations and strengths of some n point targets in 2-D with far field scattering measurements

FM analysis vs Time reversal


Exact reconstruction by FM

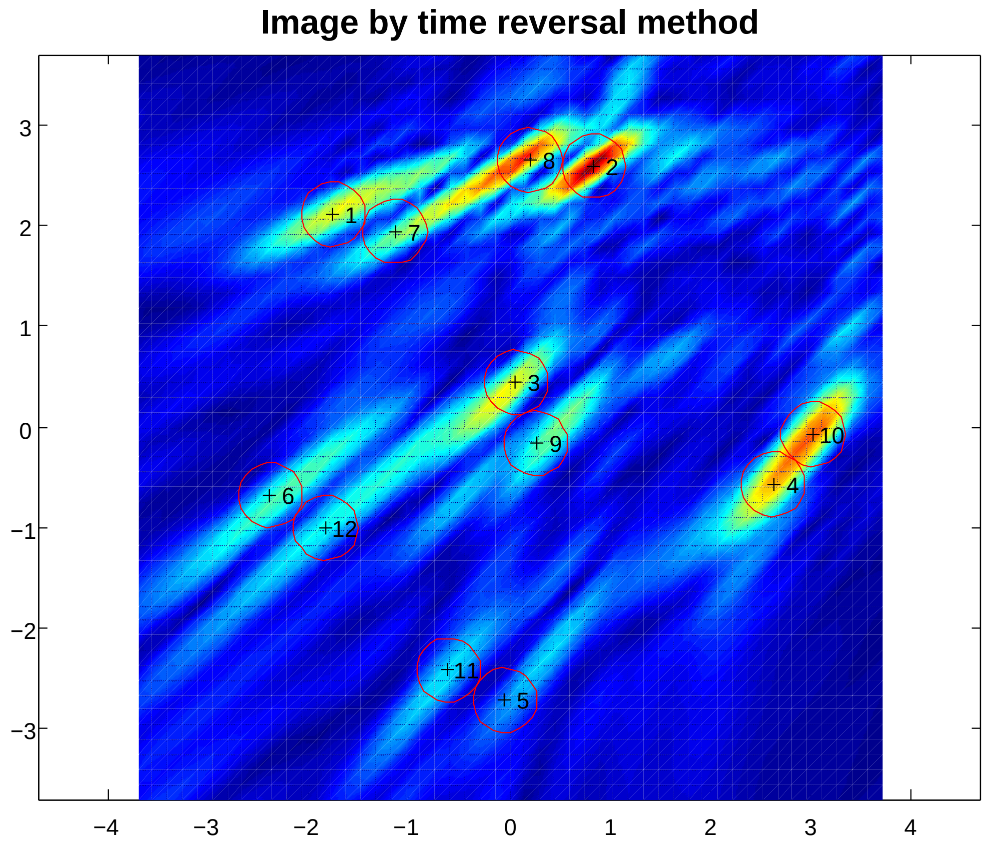


Image by time reversal method

- ♢ Point spread function, focusing: Typical AM concept
- ♢ Need to rethink about them

- ♢ FM: $O(n^3)$ steps to image n point targets

Ok, FM works well for point targets,

does it work for continuum problems ?

Yes, it does: FM to design Gaussian quadratures

Yes, it does: FM to handle Gibbs phenomenon

Gaussian quadratures for bandlimited functions

♢ Quadrature with predetermined nodes is typical AM
♢ Gaussian quadrature can be designed by FM analysis

♢ Given bandwidth b, bandlimited function $g(x)$, $x \in [-\pi, \pi]$ is generated by pure tunes $e^{ix\xi}$: $g(x) = \int_{-b}^{b} e^{ix\xi}\, c(\xi)\, d\xi$

♢ Generalized Gaussian quadratures : Select best nodes and weights to integrate each $g(x) = e^{ix\xi}$, $\xi \in [-b, b]$, to a prescribed precision

Gaussian quadratures for bandlimited functions

◊ Gaussian quadratures are useful in their own right

◊ Also closely related to imaging and signal processing

◊ FM allows for arbitrary weights and domains

◊ Quadratures can be constructed routinely in 1, 2, 3-D

Quadrature design as approximation problem

♢ The Problem

Approximate the integrals $s(k) = \int_{-\pi}^{\pi} w(x)\, e^{ikx} dx,\ k \in [-b, b]$ by a quadrature

$$s(k) \approx \sum_{j=1}^{n} w_j\, e^{ikx_j}$$

♢ Easy Solution

This merely requires FM analysis on signal $s(k)$ to figure out its frequencies x_j and amplitudes w_j

Gaussian quadrature for bandlimited functions

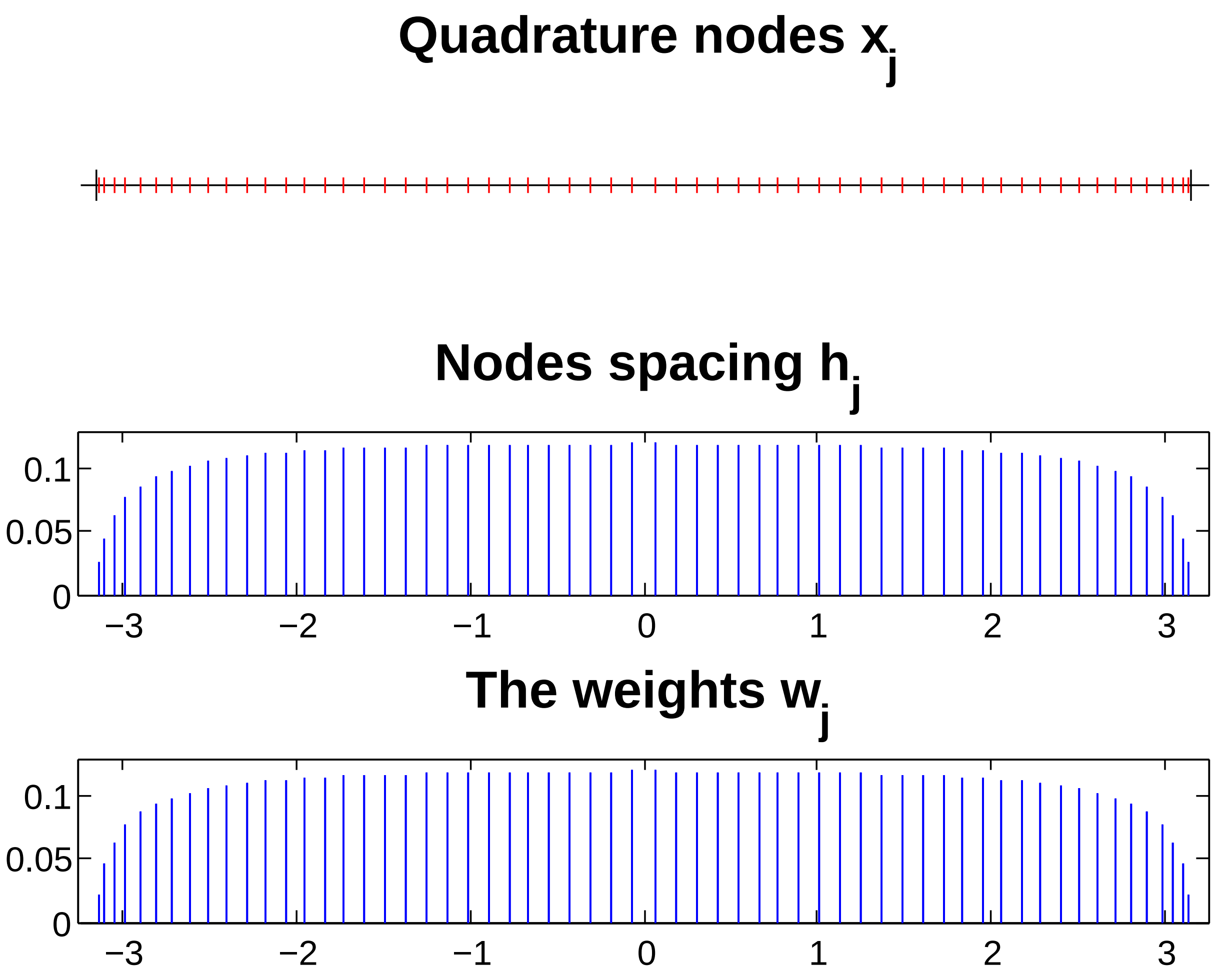
◇ Example: $w(x) = 1$, $s(k) = \int_{-\pi}^{\pi} e^{ikx} dx = \text{sinc}(k)$,

Bandwidth $b = 150/\pi$, Precision $\epsilon = \pi\ 10^{-10}$

◇ HSVD constructed 60 Gaussian nodes

◇ Quadrature error: 1.90404×10^{-11}

Gaussian quadrature for Bandlimited Functions


Quadrature nodes x_j

Nodes spacing h_j

The weights w_j

◇ Gaussian quadrature too efficient to let its weights oscillate

◇ Recover $w(x)$ by Gaussian quadrature: $w_j/h_j \approx 1 = w(x_j)$

Gaussian-Hermite quadrature for bandlimited functions

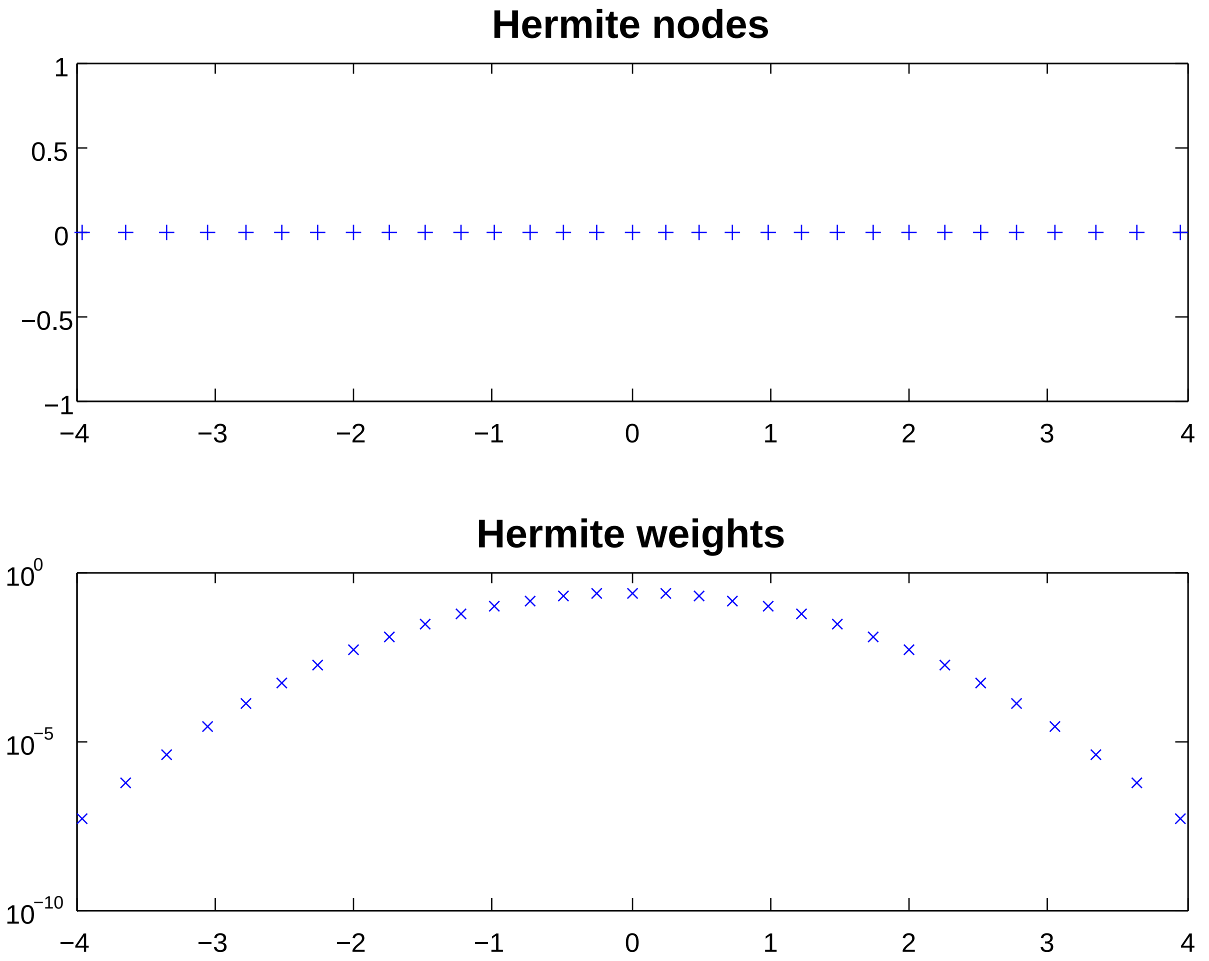
◇ Example: $w(x) = e^{-x^2}$, $s(k) = \int_{-\infty}^{\infty} e^{-x^2} e^{ikx} dx = \sqrt{\pi} e^{-k^2/4}$

Bandwidth $b = 20$: $k \in [-b, b]$.
Prescribed precision $\epsilon = 10^{-8}$

◇ HSVD constructed 31 Gaussian-Hermite nodes

◇ Quadrature error: 1.06149×10^{-8}

Gaussian Hermite Nodes for bandlimited functions



2-D Gaussian quadratures

- ♢ Useful for imaging, as well as numerical integration
- ♢ Lots of ambiguities in theory and computation
- ♢ Very few results, analytical or numerical

- ♢ I. P. Mysovskikh's results in 1970s
  - Integrate bivariate polynomials of deg $<$ 2n on domain D
  - There are n+1 orthogonal bivariate polynomials of degree n
  - They have at most n(n+1)/2 p.d. common real zeroes
  - These zeroes, if they exist, are the Gaussian nodes

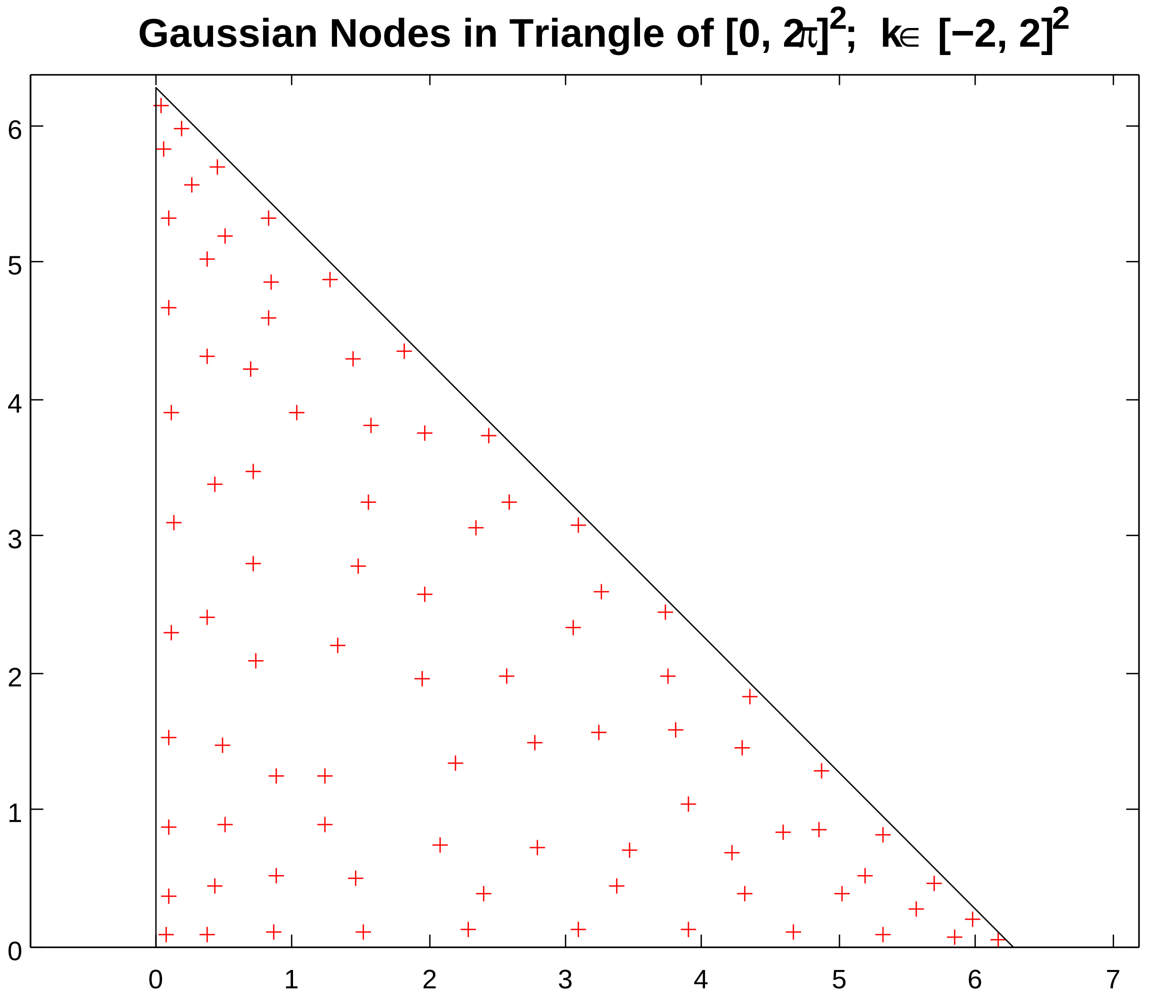
2-D Gaussian quadratures for bandlimited functions

- ♢ **Example**
  Consider all pure tunes $g(x, y) = e^{i(ux+vy)}$, $-b \leq u, v \leq b$.
  Construct G-quadrature to integrate them in a triangle
  Prescribed precision : $\epsilon = 10^{-8}$

- ♢ For $b = 2$, 2-D HSVD constructed 87 Gaussian nodes

- ♢ Quadrature error: 2.17703×10^{-7}

Gaussian nodes in triangle for bandlimited functions



FM: Able to construct Gaussian quadratures for arbitrary dimension, domain shape, and weight function

Quadrature design process interpreted as imaging process

Design Gaussian quadratures to remove Gibbs phenomenon

FM analysis to remove Gibbs phenomenon

♢ Gibbs ringing is a major difficulty in image reconstruction
... one of main sources of error in fMRI

♢ Gibbs phenomenon: Failure of AM to extract features of image from its Fourier data

♢ Traditional approaches rely on high frequency analysis
... don't work for real data : Low S/N ratio

♢ Propose three FM methods to tackle the problem

Method I: Design a Gaussian quadrature to locate jumps

Let $f(x)$ be a piecewise smooth function with jumps

♢ The Problem, in 1-D

Given Fourier data $s(k) = \int_{-\pi}^{\pi} e^{-ikx}\, f(x)\, dx$, $k = -k_0 : k_0$, design a Gaussian quadrature for e^{-ikx} with $f'(x)$ as the weight

♢ The Claim

Some quadrature nodes must fall on jump locations

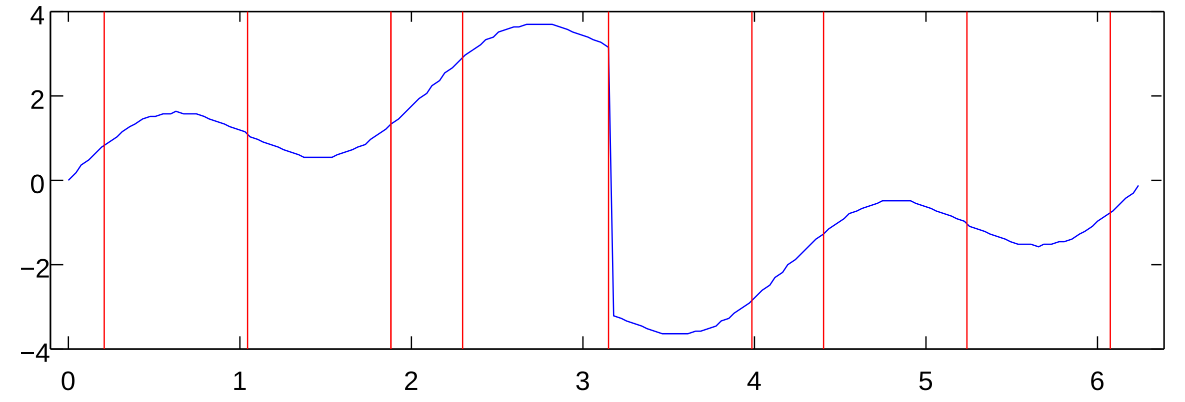
♢ The Solution

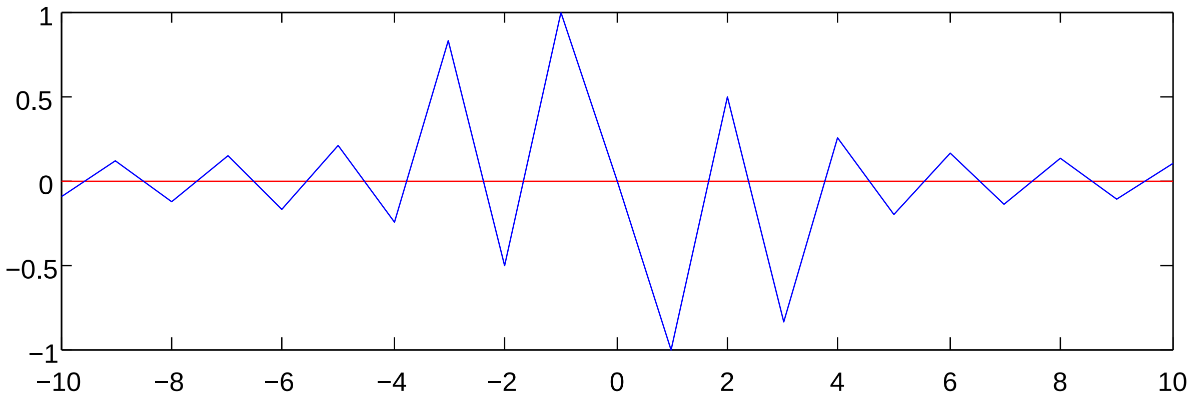
Fit $ik\, s(k) = \int_{-\pi}^{\pi} e^{-ikx}\, f'(x)\, dx$ by sum of exponentials

$$ik\, s(k) \approx \sum_j w_j\, e^{-ik\, x_j}$$

with Gaussian nodes x_j and weights w_j

Method I, Example: Gaussian quadrature to locate jumps


exact image


Fourier coefficients

◇ One of quadrature nodes falls on jump location

Method I: How it works

♢ We design a Gaussian quadrature

$$\int_{-\pi}^{\pi} f'(x)\, e^{-ikx} dx \approx \sum_j w_j\, e^{-ik\, x_j}, \qquad k \in [-k_0, k_0]$$

At the jump $x = x_\star$, $f'(x) = a\,\delta(x - x_\star) + f_0'(x)$, so

$$\int_{-\pi}^{\pi} f'(x)\, e^{-ikx} dx = a\, e^{-ikx_\star} + \int_{-\pi}^{\pi} f_0'(x)\, e^{-ikx} dx$$

Thus, the term $a\, e^{-ikx_\star}$ is a direct contribution to quadrature,
and should appear in the quadrature: $x_\star = x_l$, $a = w_l$ for some l

Method I: How it works

◇ Some of the quadrature nodes are locations of jumps

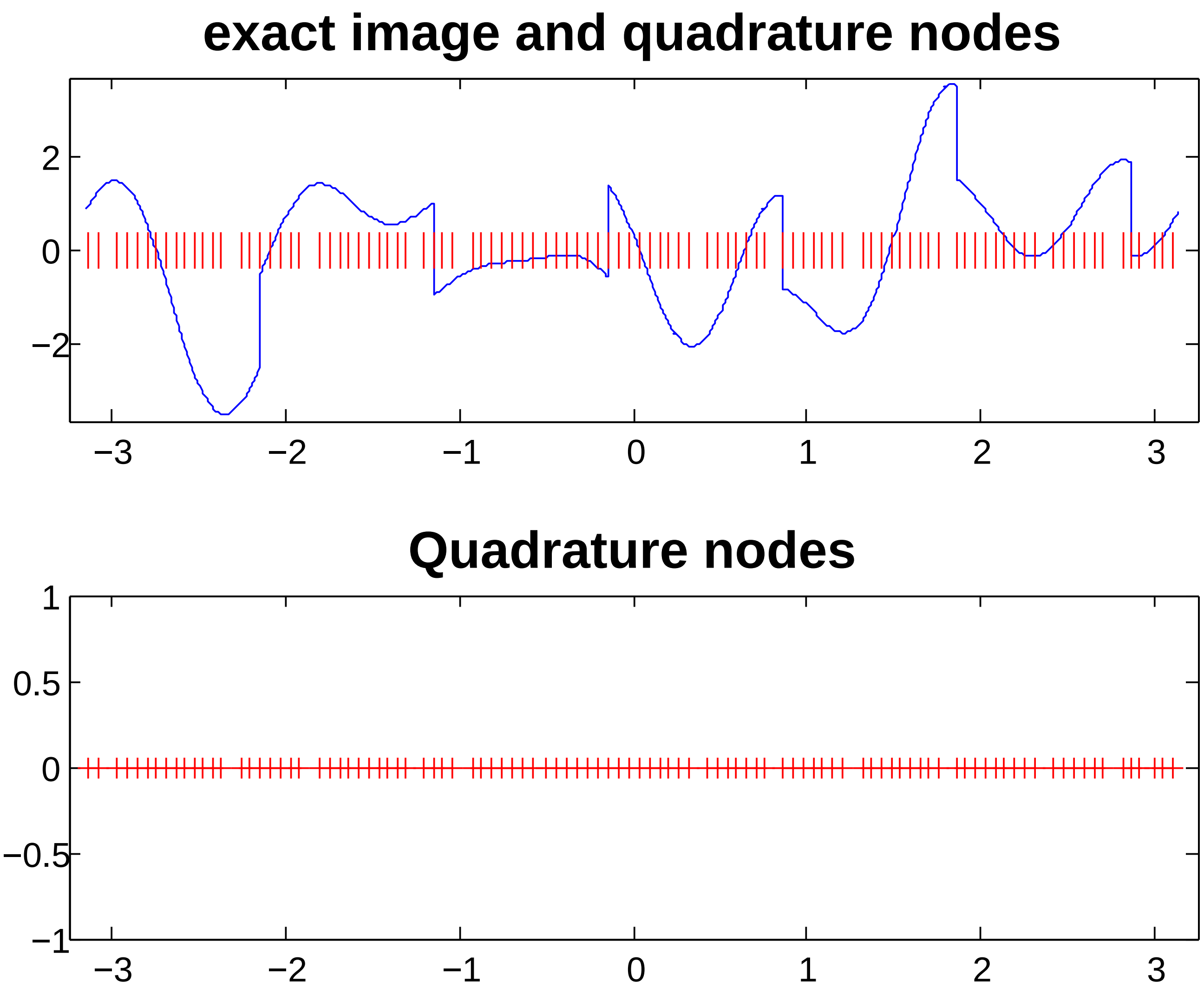
◇ The corresponding weights are magnitude of jumps

◇ Once the jumps are detected accurately, their Fourier spectrum can be removed from the data $s(k)$

◇ FFT $s(k)$ to recover the continuous part of f(x)

◇ Not clear how to extend it to higher dimensions

Method I: Locate multiple jumps



◇ Some quadrature nodes lie on jump locations

Method II : Design Gaussian quadrature to reconstruct f

♢ This is a FM method that can be extended to 2-D

♢ Design a Gaussian quadrature for bandlimited functions e^{-ikx} with $f(x)$ as the weight function

$$s(k) = \int_{-\pi}^{\pi} e^{-ikx} f(x) dx \approx \sum_j w_j \, e^{-ik\,x_j}, \quad k = -k_0 : k_0$$

... namely to fit the signal $s(k)$ by sum of exponentials with Gaussian nodes x_j and weights w_j

Method II : Design Gaussian quadrature to reconstruct f

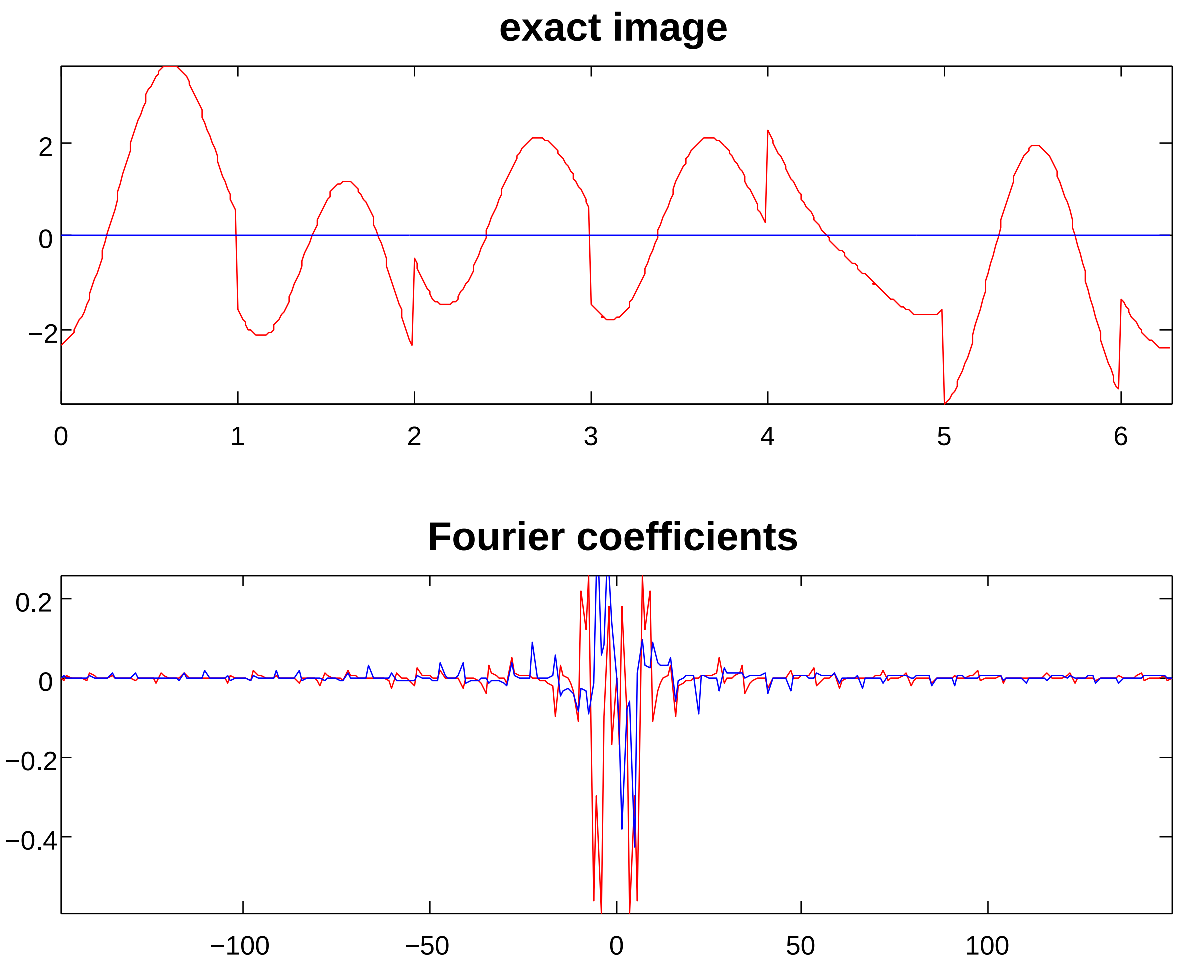
$$\int_{-\pi}^{\pi} e^{-ikx} f(x)dx \approx \sum_j w_j \, e^{-ik\, x_j}$$

♢ Weak-star convergence: $f(x_j) \approx w_j/h_j$

♢ Since Gaussian quadrature is very efficient, its weights w_j are not expected to oscillate where f jumps

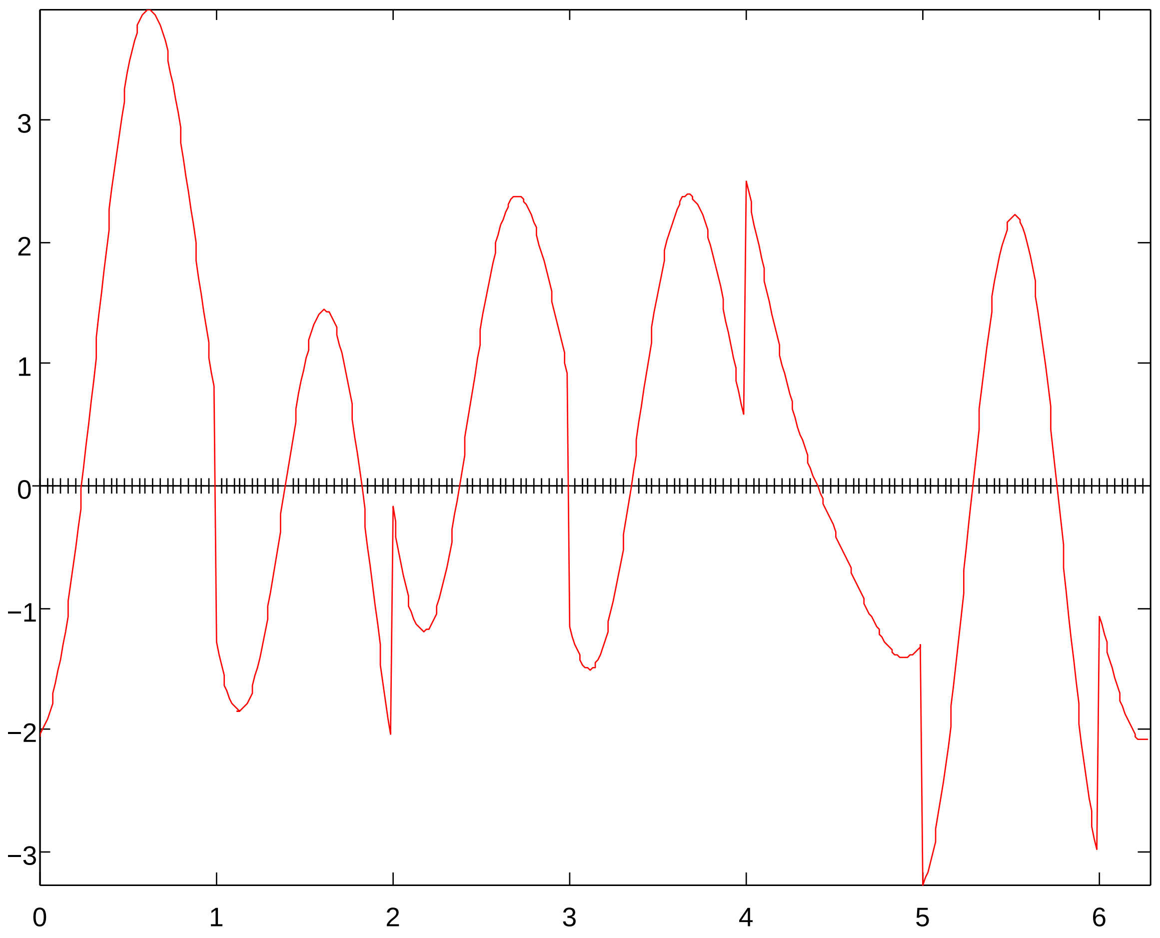
♢ Since we can design Gaussian quadrature for 2-D, this method is generalizable to higher dimensions

Method II - Example



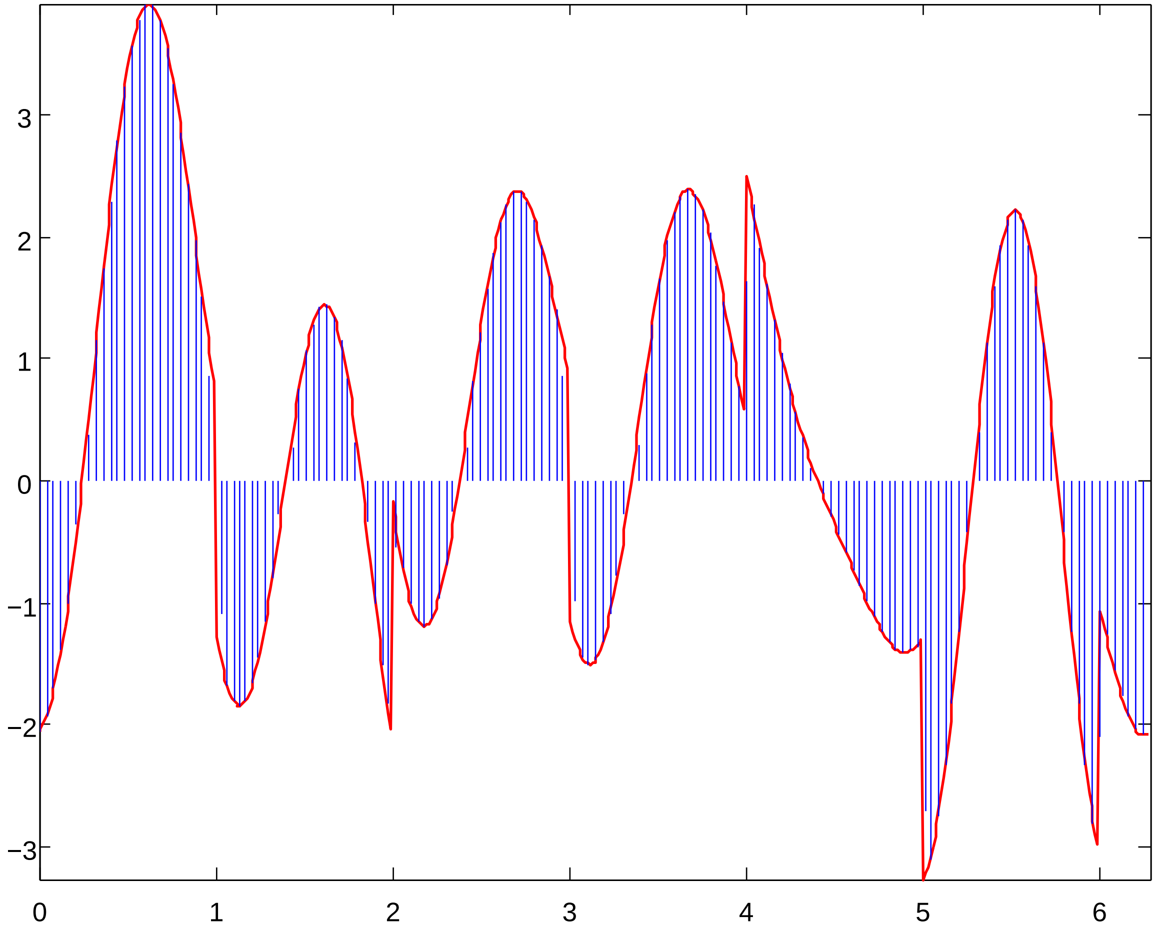
Method II - Example

The quadrature nodes

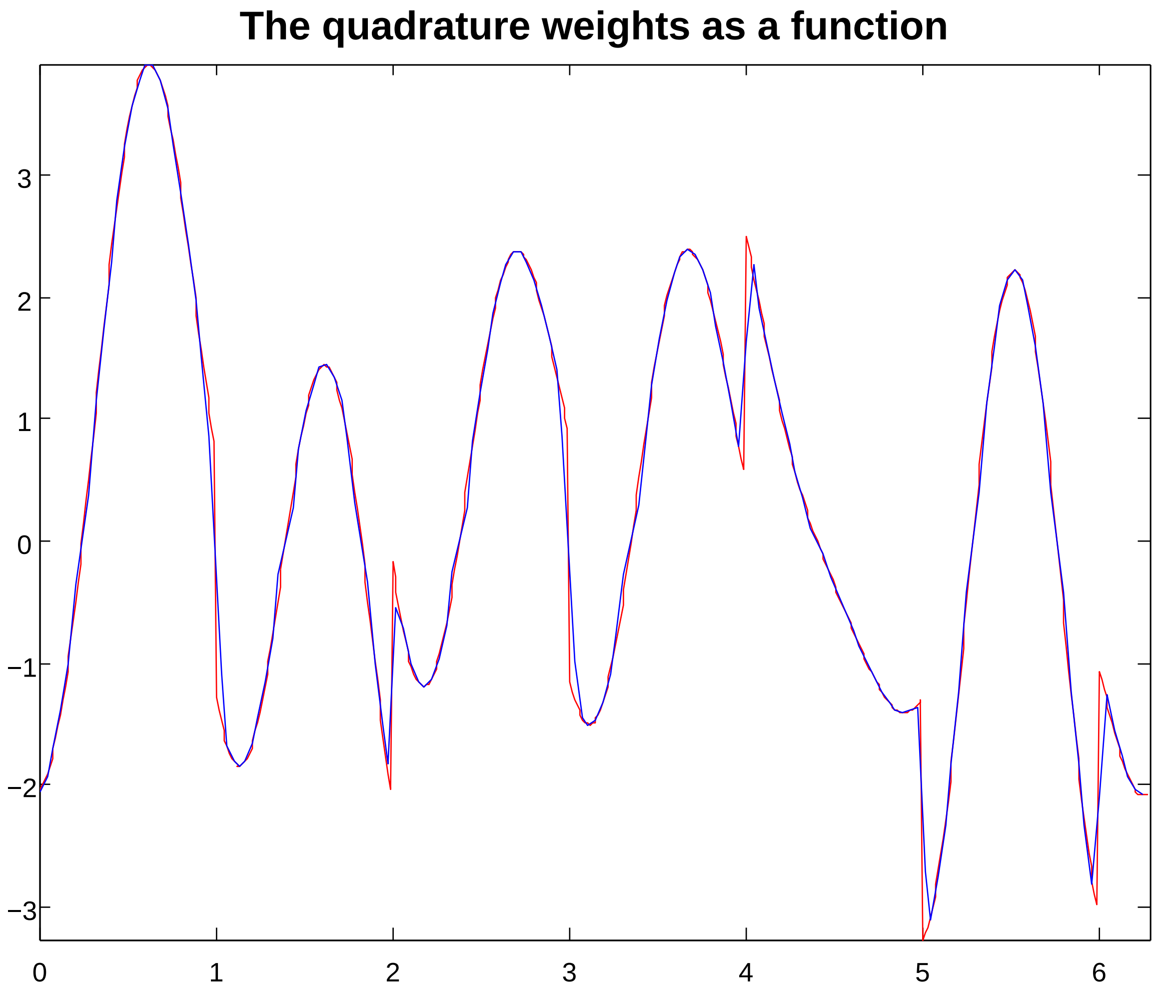


Method II - Example

The quadrature weights, properly scaled



Method II - Example



◇ Weak-star convergence : Accuracy near jumps will improve significantly using fact that image is piecewise smooth

Method III : Use poles to reconstruct f

$\diamondsuit$ Sum of poles, or rational functions, are powerful tools to approximate smooth, discontinuous, and singular functions very well

$\diamondsuit$ It is well established that rational functions approximate jumps with exponential rate of convergence

Fitting a discontinuous $f(x)$ by sum of poles

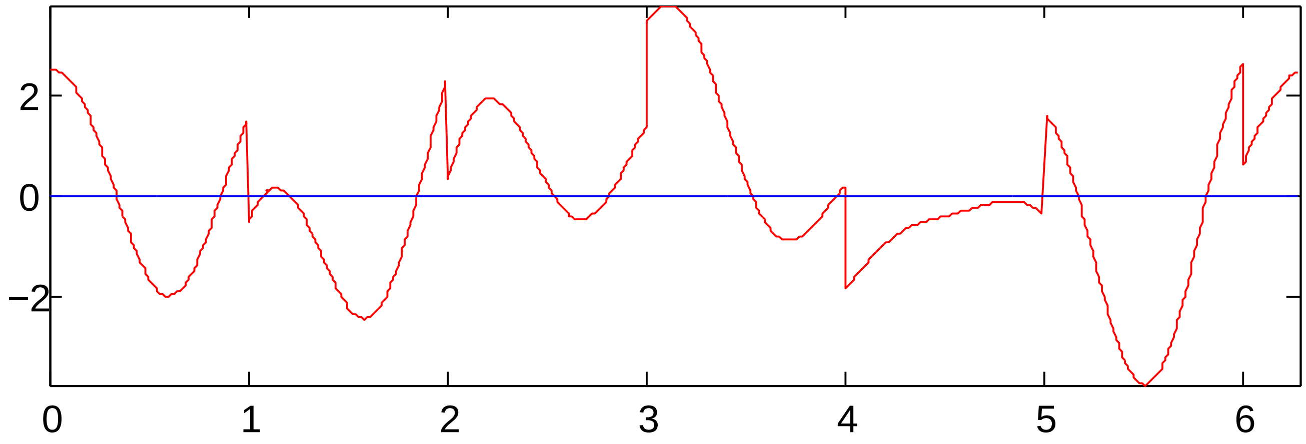
is the same as

Fitting the signal $s(k)$ by sum of exponentials

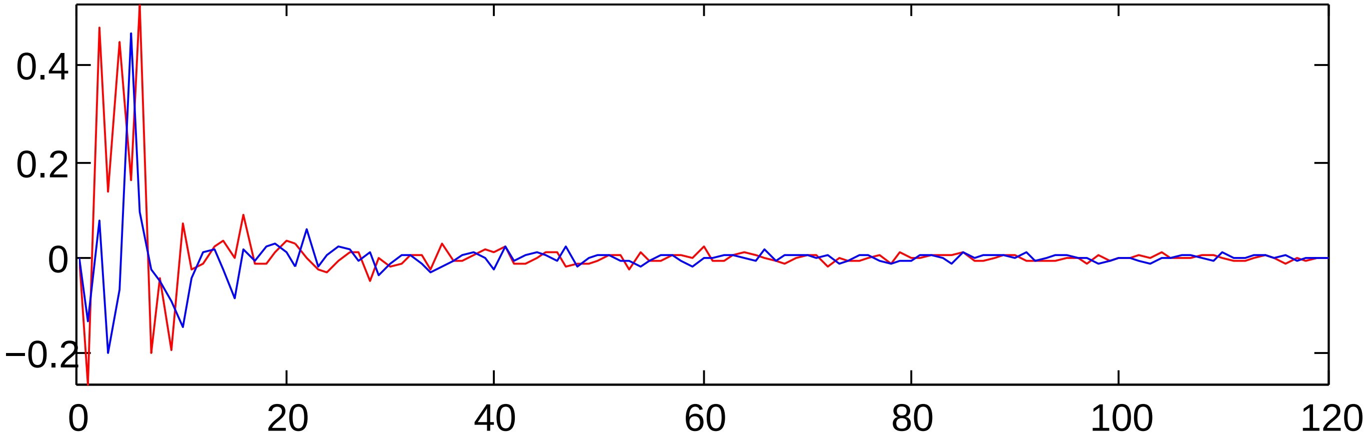
$\diamondsuit$ Not clear how to generalize to higher dimensions

Method III - Example

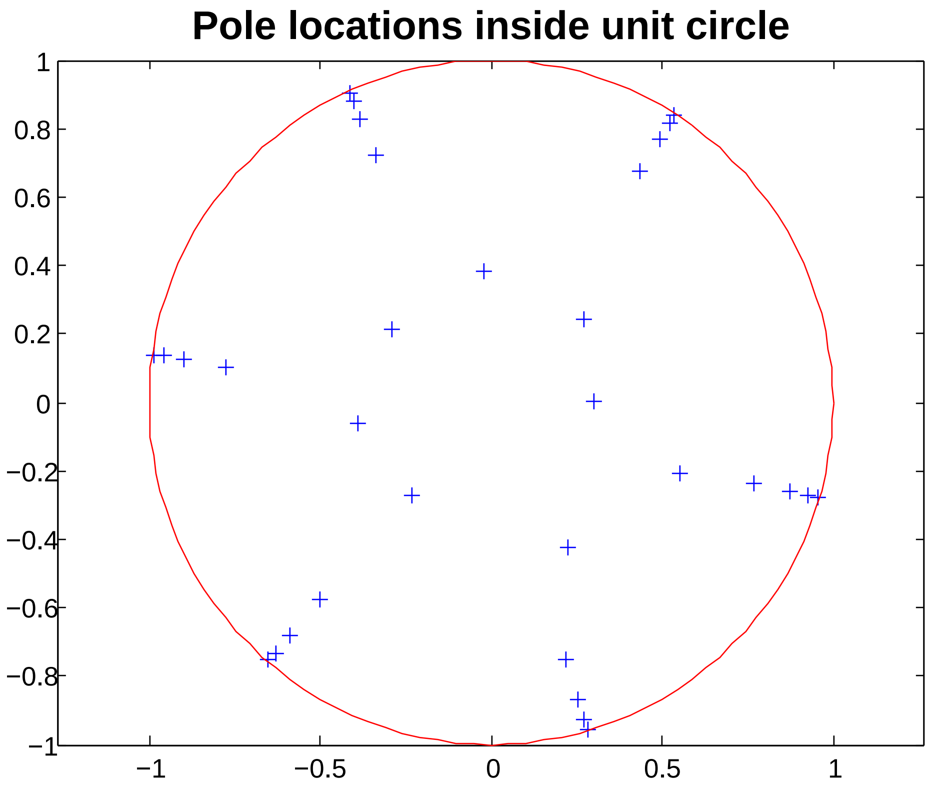
exact image



onesided Fourier coefficients

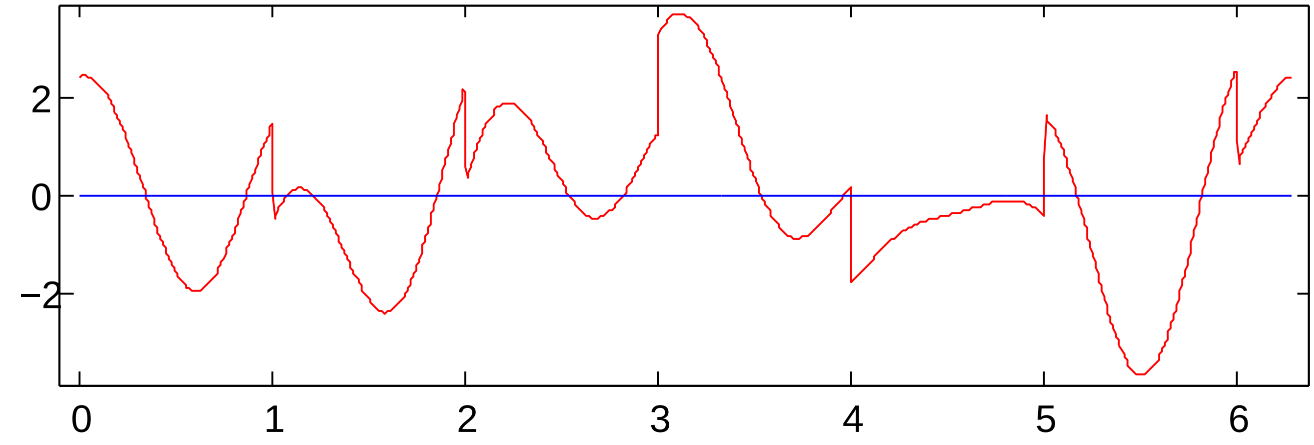


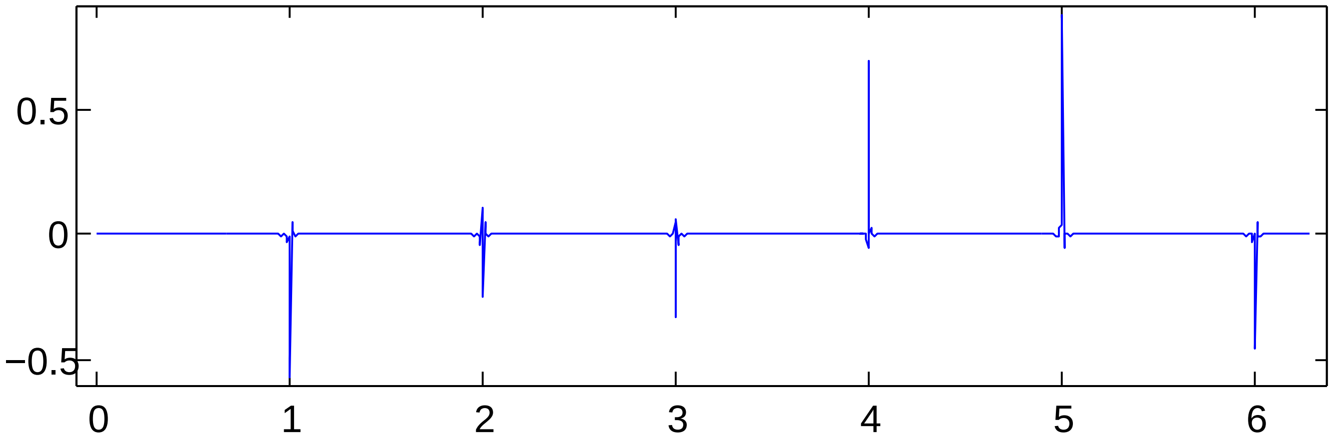
Method III - Example



- ♢ $f(x)$, $0 \leq x \leq 2\pi$, is regarded as defined on unit circle
- ♢ Poles approaching jump locations on unit circle

Method III - Example


reconstructed image


error in reconstruction

Method III

- The reconstruction is very clean and sharp

- Poles are very flexible basis, pole fitting is nonlinear

- Sum of Poles is as fundamental to sensing and imaging as sum of exponential

- Knowledge of rational functions very incomplete in 2-D

Conclusion: the Fourier transform

◇ Fourier transform extremely useful for sensing and imaging

◇ It works if overwhelming data are available

◇ Inadequate to recover features of image if data are sufficient, but not overwhelming

... think about 4 $\sim$ 10 fold reduction of MRI data
... or mammography with cm EM wave and limited aperture
... or perhaps over-the-horizon, low-frequency radar (Stealth?)

- All these may be done with higher resolution by FM

Conclusion: Nonlinearity and FM

♢ Extracting feature efficiently requires nonlinear processing

♢ FM seems to do it seamlessly: No grid or mesh required in the physical domain

♢ FM analysis systematically explores the phase information

- Higher resolution in imaging
- No need to deal with focusing, point spread function
- Ideal for imaging point targets
- Capable of extracting features in image
- Works well for limited aperture

Conclusion and Discussion

- ♢ AM/FM imaging is similar to AM/FM radio
- ♢ The phase encodes more information than amplitude
- ♢ Phase information is more stable, hard to corrupt by noise
- ♢ FM analysis systematically exploits phase information

Solve simultaneous system of equations

- Inverse scattering: Solve system of equations all the time

- Imaging: Hardly solve any equations

  - We use Fourier transforms a lot
  - We rely on pulsed beams when possible

  But when we have limited aperture data, we don't know how to solve under determined system of equations

- "Gaussian Elimination" is most under-used in imaging

- FM analysis: Reliable way to deal with it systematically

Existing technology

◊ Pulsed-beam Radar, Sonar, Doppler: Severe requirements on power and frequency : high frequency, high power, low resolution, long data acquisition time, expensive sensor arrays

◊ Point spread function, spurious Fresnel zone, side lobes, focusing, lenses : Old AM concept based on simple geometry considerations, not “Gaussian Elimination”

◊ SAR, interferometry still done with AM. Phase information not exploited efficiently by “Gaussian Elimination”

◊ FM may be helpful for these applications

Low frequency Radar and Sonar

◇ FM analysis allows for continuous, low power, low frequency waves with much higher resolutions

◇ No need for beamforming, greater clutter reduction

◇ Low frequency Radar: 3-30 MHz, 100-10 meter wavelength

◇ Low frequency Sonar : 100 Hz, longer range, higher S/N ratio

◇ Lower frequency reconnaissance Radar to penetrate foliage

New opportunities for FM analysis

♢ 2-D FM analysis : Stability to analyze, fast algorithms wanted

♢ FM analysis for nonlinear inverse scattering problems

♢ Forward scattering, general purpose ray tracing by FM (Euler's method for highly oscillatory problems)

♢ Signal processing by FM, Time frequency representation

Whenever Fourier analysis is used, there is opportunity to reformulate it with FM

♢ A lot to be understood : Theory, algorithm, efficiency, stability

Least norm solutions in L^0, L^1, L^2

♢ Between AM and FM is Compressed Sensing (CS) based on L^1 minimization

♢ It seems that the L^1 approach is not specific enough for wave related sensing and imaging applications

♢ AM : Least 2-norm solution
♢ CS : Least 1-norm solution
♢ FM : Least 0-norm solution - the sparsest solution

Regularization is currently done with 2-norm, but should it be done with 0-norm for imaging and inverse scattering ?