High Order Numerical Solution of the Nonlinear Helmholtz Equation with Material Discontinuities

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Plan of Presentation

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1. Formulation of the problem
   - Background
   - Differential equation and boundary conditions
   - Prior work and current objectives
2. Numerical method
   - Integral formulation
   - Second and fourth order compact schemes
   - Two-way boundary conditions
   - Newton's iterations
3. Results of computations
   - Convergence of iterations
   - Computational error
4. Discussion
   - Summary
   - Possible future extensions

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- Full nonlinear Maxwell equations.
  - Single frequency (time-harmonic) approximation.
- Vector nonlinear Helmholtz equation.
  - Linear polarization.
- Scalar nonlinear Helmholtz equation (NLH).
  - Paraxial (parabolic) approximation.
- Scalar nonlinear Schrödinger equation (NLS).

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- **Key physical phenomenon of interest:** nonlinear self-focusing.

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Helmholtz equation and Schrödinger equation

- Kerr medium, n_0 — linear refraction index, n_2 — Kerr coefficient:

$$\Delta E(\boldsymbol{x}) + k_0^2(1+\epsilon|E|^{2\sigma})E = 0, \quad k_0 = \frac{\omega_0}{c} n_0^{\text{ext}}, \quad \epsilon = 2n_2 n_0/(n_0^{\text{ext}})^2.$$

- Let z be the direction of propagation and r_0 be the initial width of the beam. Change the variables and introduce the ansatz for E:

$$\tilde{x}_\perp = \frac{x_\perp}{r_0}, \quad \tilde{z} = \frac{z}{2k_0 r_0^2}, \quad E = e^{ik_0 z}(\epsilon r_0^2 k_0^2)^{-1/2\sigma}\psi(\tilde{x}_\perp, \tilde{z}).$$

- In the new variables, the NLH transforms into

$$i\psi_{\tilde{z}}(\tilde{\boldsymbol{x}}, \tilde{z}) + \Delta_\perp \psi + |\psi|^{2\sigma}\psi = -4f^2\psi_{\tilde{z}\tilde{z}},$$

where $f = 1/r_0 k_0 \ll 1$ is the nonparaxiality parameter.

- Paraxial approximation yields the NLS by dropping the $\mathcal{O}(f^2)$ term:

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Schrödinger equation vs. Helmholtz equation

- The nonlinear Schrödinger equation (NLS):
  - Requires initial conditions — evolution problem.
  - Only forward propagation — no backscattering.
  - Solution blow-up (collapse) occurs for strong nonlinearities, which, however, is not observed in the actual experiments.
  - Paraxial approximation breaks down near singularity: [Kelly '65].
- The nonlinear Helmholtz equation (NLH):
  - Requires boundary conditions — elliptic problem.
  - Propagation in all directions.
  - Well studied in 1D, but little is known in multi-D.

Important scientific question:

Does the NLH have singular solutions **OR** it helps arrest the collapse?

- Nonparaxiality may indeed remove the singularity: [Feit & Fleck '88], [Fibich '96], but it has never been for the true NLH.

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What is known about the multi-D NLH?

- Very recent solvability results for real Robin boundary conditions (not Sommerfeld); nonuniqueness: [Sever '06].
- Numerical solution: [Fibich & Tsynkov '01, '05].
  - Homogeneous case: $n_0 = \text{const}$ & $n_2 = \text{const}$.
  - Fourth order central-difference scheme on a five-node stencil.
  - Iterative solution by freezing the nonlinearity and then using the sequence of Born approximations.
- The method has enabled computation of some interesting cases (near critical focusing, narrow spatial solitons, etc.)
- Shortcomings of the method:
  - Loses accuracy (reduces to second order) on fine grids because of the discontinuities at the boundaries due to nonlinearity.
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Methodological question:

Is the breakdown of convergence to be attributed to the properties of the solution (e.g., nonuniqueness) or to the deficiencies of the solver?

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The one-dimensional NLH

- Assume that all the quantities may vary only in one direction — the direction of propagation z: $E = E(z)$, $n_0 = n_0(z)$, $n_2 = n_2(z)$.

- 1D NLH, normalized coefficients:

$$\frac{d^2E(z)}{dz^2} + k_0^2\left(\nu(z) + \epsilon(z)\,|E|^2\right)E = 0,$$

$$\nu = (n_0/n_0^{\text{ext}})^2, \quad \epsilon = 2n_2 n_0/(n_0^{\text{ext}})^2,$$

$\nu \equiv 1$ & $\epsilon \equiv 0$ for $z < 0$ & $z > Z_{\max}$.

- The medium is layered: $0 = \tilde{z}_1 < \cdots < \tilde{z}_l < \cdots < \tilde{z}_L = Z_{\max}$,

$\nu(z) \equiv \tilde{\nu}_{l+\frac{1}{2}}$ and $\epsilon(z) \equiv \tilde{\epsilon}_{l+\frac{1}{2}}$ for $z \in (\tilde{z}_l, \tilde{z}_{l+1})$.

- Discontinuities at the outer boundaries are always there, even if the interior Kerr medium is homogeneous.
- $E(z)$ and $\frac{dE}{dz}$ are supposed to be continuous across every interface.
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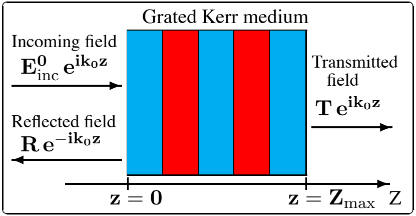

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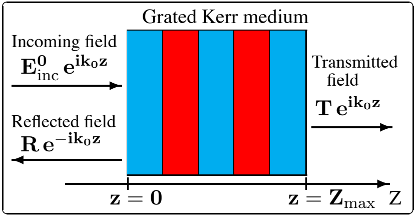

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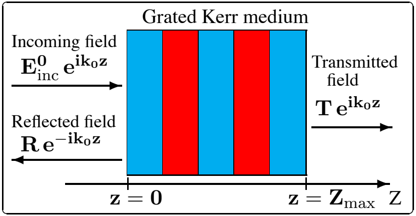

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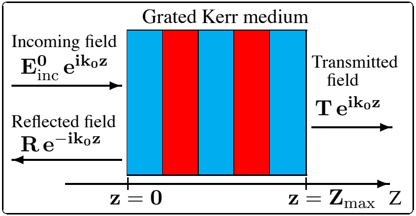

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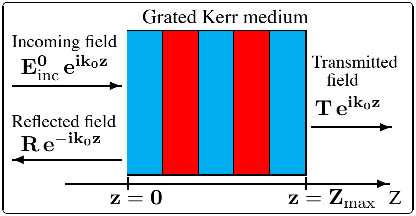

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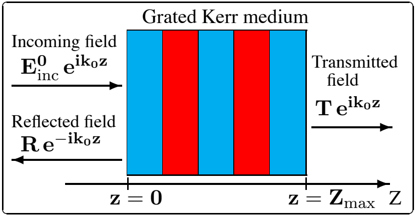
- Assume that all the quantities may vary only in one direction — the direction of propagation z: $E = E(z)$, $n_0 = n_0(z)$, $n_2 = n_2(z)$.



- 1D NLH, normalized coefficients:

$$\frac{d^2E(z)}{dz^2} + k_0^2\left(\nu(z) + \epsilon(z)\,|E|^2\right)E = 0,$$

$$\nu = (n_0/n_0^{\text{ext}})^2, \quad \epsilon = 2n_2 n_0/(n_0^{\text{ext}})^2,$$

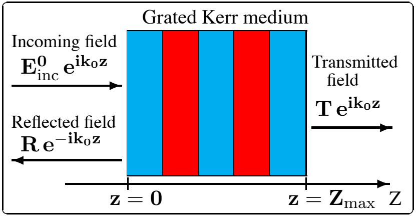
$$\nu \equiv 1 \;\&\; \epsilon \equiv 0 \text{ for } z < 0 \;\&\; z > Z_{\text{max}}.$$

- The medium is layered: $0 = \tilde{z}_1 < \cdots < \tilde{z}_l < \cdots < \tilde{z}_L = Z_{\text{max}}$,

$$\nu(z) \equiv \tilde{\nu}_{l+\frac{1}{2}} \text{ and } \epsilon(z) \equiv \tilde{\epsilon}_{l+\frac{1}{2}} \quad \text{for} \quad z \in (\tilde{z}_l, \tilde{z}_{l+1}).$$

- Discontinuities at the outer boundaries are always there, even if the interior Kerr medium is homogeneous.
- $E(z)$ and $\frac{dE}{dz}$ are supposed to be continuous across every interface.
- The medium is assumed lossless: $n_0, n_2 \in \mathbb{R} \Rightarrow \nu, \epsilon \in \mathbb{R}$.

Boundary conditions



- Incoming+reflected field for $z<0$:

$$E(z) = E^0_{\rm inc}e^{ik_0z} + Re^{-ik_0z}.$$

- Transmitted field for $z>Z_{\max}$:

$$E(z) = Te^{ik_0z}.$$

- The boundary conditions at $z=0$ and $z=Z_{\max}$ are obtained using the continuity of E and $\frac{dE}{dz}$.
- Transparent BC at $z=Z_{\max}$: $\left(\frac{d}{dz}-ik_0\right)E\Big|_{z=Z_{\max}} = 0.$
- Two-way BC at $z=0$: $\left(\frac{d}{dz}+ik_0\right)E\Big|_{z=0} = 2ik_0E^0_{\rm inc}.$
- Rescaling: $E\mapsto E/E^0_{\rm inc}$, $\epsilon\mapsto\epsilon|E^0_{\rm inc}|^2$, so that $E^0_{\rm inc}=1$ and the variation in ϵ is equivalent to the variation in the input power.

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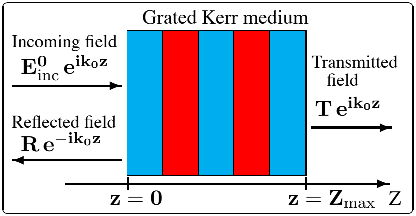

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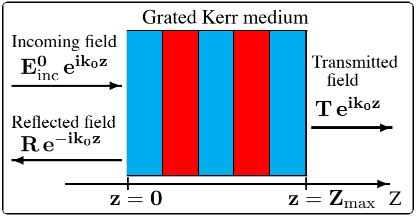

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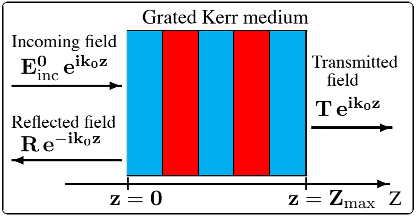

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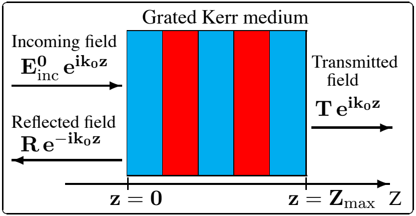

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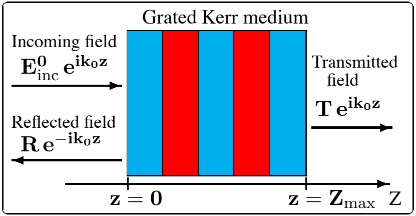

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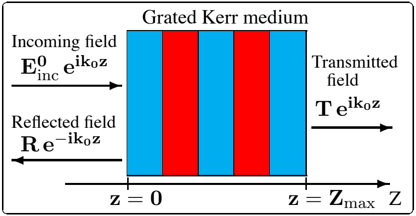

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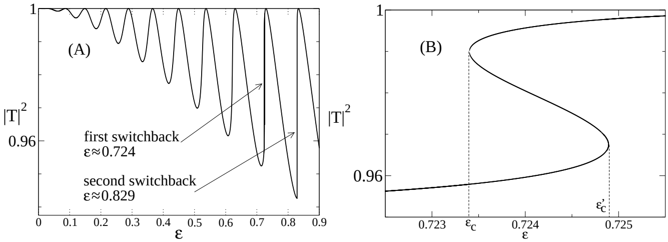
Prior work on 1D NLH

- Closed form exact solutions: [Wilhelm '70], [Marburger & Felber '78], [Chen & Mills '87]. The solution is not unique.



- Finite element numerical method: [Suryanto, et al. '02, '03].
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Answer to the methodological question:

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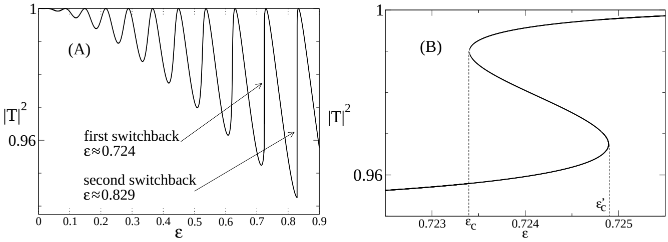

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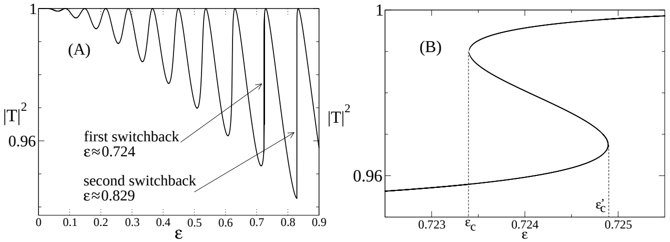

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Important observation 1

Frozen nonlinearity iterations stop converging for large ϵ (even in 1D).

Specific objective 1

To build an iteration scheme that would handle strong nonlinearities.

Important observation 2

$E''(z)$ and higher order derivatives are discontinuous at interfaces.

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To design a scheme for the 1D NLH that would maintain high order accuracy across the entire domain, including the discontinuities.

Long-term objective

To attain similar computational capabilities for the multi-D NLH.

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Integral formulation

- Let $a,\ b \in [0, Z_{\max}]$; integrate the NLH and use the continuity of E':

$$\frac{dE(b)}{dz} - \frac{dE(a)}{dz} + k_0^2 \int_a^b \left(\nu(z) + \epsilon(z)\,|E|^2\right) E\,dz = 0.$$

- The differential and integral formulations are equivalent for smooth solutions. The latter remains valid for discontinuous $E''(z)$ as well.
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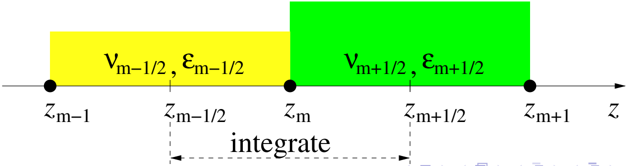

Integral formulation (cont'd)

- **It is now the integral relation:**

$$\left.\frac{dE}{dz}\right|_{z_{m-\frac{1}{2}}}^{z_{m+\frac{1}{2}}} + k_0^2\nu_{m-\frac{1}{2}} \int_{z_{m-\frac{1}{2}}}^{z_m} E\,dz + k_0^2\epsilon_{m-\frac{1}{2}} \int_{z_{m-\frac{1}{2}}}^{z_m} |E|^2 E\,dz$$
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- $E(z)$ is infinitely differentiable within each cell, and we can employ polynomial interpolation to approximate the integrals and fluxes.
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- On the regions of smoothness, the resulting finite volume scheme will be equivalent to compact finite differences.

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Second order scheme

- Approximation of the flux difference:

$$\frac{dE}{dz}\Big|_{z_{m-\frac{1}{2}}}^{z_{m+\frac{1}{2}}} = \frac{E_{m+1}-E_m}{h} - \frac{E_m - E_{m-1}}{h} + \mathcal{O}\left(h^2\right).$$

- Approximation of the integrals, $z \in [z_{m-\frac{1}{2}}, z_{m+\frac{1}{2}}]$, $\zeta \in [0, 1/2]$:

$$E\left(z_m + h\zeta\right) = \underbrace{(1-\zeta)}_{F_0(\zeta)} E_m + \underbrace{\zeta}_{F_1(\zeta)} E_{m+1} + \mathcal{O}\left(h^2\right),$$

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Second order scheme (cont'd)

- Other integrals are approximated similarly, which altogether yields:

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- On "smooth" regions, $\nu_{m-\frac{1}{2}} = \nu_{m+\frac{1}{2}} = \nu$ and $\epsilon_{m-\frac{1}{2}} = \epsilon_{m+\frac{1}{2}} = \epsilon$, we obtain a second order central-difference scheme for the 1D NLH:

$$\frac{E_{m+1}-2E_m+E_{m-1}}{h^2} + k_0^2\nu\sum_{i=0}^{1} f_i(E_{m-i}+E_{m+i}) + k_0^2\epsilon\sum_{i,j,k=0}^{1} g_{ijk}\left(E^*_{m-i}E_{m-j}E_{m-k}+E^*_{m+i}E_{m+j}E_{m+k}\right) = 0.$$

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- Other integrals are approximated similarly, which altogether yields:

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$$+\ hk_0^2\nu_{m-\frac{1}{2}}\sum_{i=0}^{1} f_i E_{m-i}+hk_0^2\epsilon_{m-\frac{1}{2}}\sum_{i,j,k=0}^{1} g_{ijk}E^*_{m-i}E_{m-j}E_{m-k}$$
$$+\ hk_0^2\nu_{m+\frac{1}{2}}\sum_{i=0}^{1} f_i E_{m+i}+hk_0^2\epsilon_{m+\frac{1}{2}}\sum_{i,j,k=0}^{1} g_{ijk}E^*_{m+i}E_{m+j}E_{m+k}=0.$$

- On "smooth" regions, $\nu_{m-\frac{1}{2}}=\nu_{m+\frac{1}{2}}=\nu$ and $\epsilon_{m-\frac{1}{2}}=\epsilon_{m+\frac{1}{2}}=\epsilon$, we obtain a second order central-difference scheme for the 1D NLH:

$$\frac{E_{m+1}-2E_m+E_{m-1}}{h^2}+k_0^2\nu\sum_{i=0}^{1} f_i(E_{m-i}+E_{m+i})$$
$$+\,k_0^2\epsilon\sum_{i,j,k=0}^{1} g_{ijk}\left(E^*_{m-i}E_{m-j}E_{m-k}+E^*_{m+i}E_{m+j}E_{m+k}\right)=0.$$

Fourth order scheme

- One-sided second derivatives are evaluated using the equation:

$$E''_{m+0} \stackrel{\text{def}}{=} \left.\frac{d^2E}{dz^2}\right|_{z=z_m+0} = -k_0^2\left(\nu_{m+\frac{1}{2}} + \epsilon_{m+\frac{1}{2}}\left|E_m\right|^2\right)E_m,$$

$$E''_{(m+1)-0} \stackrel{\text{def}}{=} \left.\frac{d^2E}{dz^2}\right|_{z=z_{m+1}-0} = -k_0^2\left(\nu_{m+\frac{1}{2}} + \epsilon_{m+\frac{1}{2}}\left|E_{m+1}\right|^2\right)E_{m+1}.$$

- Approximation of the fluxes:

$$\begin{aligned}
E'_{m+\frac{1}{2}} &= \frac{E_{m+1}-E_m}{h} - \frac{h^2}{24}E^{(3)}_{m+\frac{1}{2}} + \mathcal{O}\left(h^4\right) \\
&= \frac{E_{m+1}-E_m}{h} - \frac{h^2}{24}\frac{E''_{(m+1)-0}-E''_{m+0}}{h} + \mathcal{O}\left(h^4\right) \\
&= \frac{1}{h}\left(1+\frac{(k_0h)^2}{24}\left(\nu_{m+\frac{1}{2}} + \epsilon_{m+\frac{1}{2}}\left|E_{m+1}\right|^2\right)\right)E_{m+1} \\
&- \frac{1}{h}\left(1+\frac{(k_0h)^2}{24}\left(\nu_{m+\frac{1}{2}} + \epsilon_{m+\frac{1}{2}}\left|E_m\right|^2\right)\right)E_m + \mathcal{O}\left(h^4\right).
\end{aligned}$$

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Hermite-Birkhoff interpolation

Lemma

Let $E \in C^4([z_m, z_{m+1}])$. Let $E_m = E(z_m)$ and $E_{m+1} = E(z_{m+1})$ be known along with one-sided second derivatives E''_{m+0} and $E''_{(m+1)-0}$. Then, the function $E(z)$ is approximated with fourth order accuracy:

$$E(z_m + \zeta h) = P_3(\zeta) + \mathcal{O}\left(h^4\right), \qquad z \in \left[z_m, z_{m+1}\right],$$

by the Hermite-Birkhoff cubic polynomial:

$$P_3(\zeta) = \left(E_m - \frac{h^2}{6}E''_{m+0}\right)(1-\zeta) + \frac{h^2}{6}E''_{m+0}(1-\zeta)^3 + \left(E_{m+1} - \frac{h^2}{6}E''_{(m+1)-0}\right)\zeta + \frac{h^2}{6}E''_{(m+1)-0}\zeta^3.$$

Given E_m, E_{m+1}, E''_{m+0}, and $E''_{(m+1)-0}$, the polynomial P_3 is unique.

Fourth order scheme (cont'd)

- Substituting the expressions for E''_{m+0} and $E''_{(m+1)-0}$, we have:

$$E(z_m+\zeta h)=\sum_{i=0}^{3}\underbrace{F_i(\zeta,h,\nu_{m+\frac{1}{2}})}_{\text{cubic polynomials}}v_i^+ + \mathcal{O}\left(h^4\right),$$

$$v_0^+=E_m,\ v_1^+=\epsilon_{m+\frac{1}{2}}\left|E_m\right|^2E_m,\ v_2^+=E_{m+1},\ v_3^+=\epsilon_{m+\frac{1}{2}}\left|E_{m+1}\right|^2E_{m+1}.$$

- Approximation of the integrals:

$$\int_{z_m}^{z_{m+\frac{1}{2}}}E\,dz=h\sum_{i=0}^{3}\underbrace{\left(\int_0^{\frac{1}{2}}F_i(\zeta,h,\nu_{m+\frac{1}{2}})\,d\zeta\right)}_{f_i}v_i^+ + \mathcal{O}\left(h^5\right),$$

$$\int_{z_m}^{z_{m+\frac{1}{2}}}\left|E\right|^2E\,dz=h\sum_{i,j,k=0}^{3}\underbrace{\left(\int_0^{\frac{1}{2}}F_iF_jF_k\,d\zeta\right)}_{g_{ijk}}(v_i^+)^*v_j^+v_k^+ + \mathcal{O}\left(h^5\right).$$

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Fourth order scheme (cont'd)

- Everything is similar for $[z_{m-\frac{1}{2}}, z_m]$, and altogether we have:

$$\frac{E_{m+1}-E_m}{h}\left(1+\nu_{m+\frac{1}{2}}\frac{h^2k_0^2}{24}\right)-\frac{E_m-E_{m-1}}{h}\left(1+\nu_{m-\frac{1}{2}}\frac{h^2k_0^2}{24}\right)$$
$$+\frac{h^2k_0^2}{24}\left(\epsilon_{m+\frac{1}{2}}\frac{|E_{m+1}|^2E_{m+1}-|E_m|^2E_m}{h}-\epsilon_{m-\frac{1}{2}}\frac{|E_m|^2E_m-|E_{m-1}|^2E_{m-1}}{h}\right)$$
$$+hk_0^2\nu_{m-\frac{1}{2}}\sum_{i=0}^{3}f_i(\nu_{m-\frac{1}{2}})v_i^- + hk_0^2\epsilon_{m-\frac{1}{2}}\sum_{i,j,k=0}^{3}g_{ijk}(\nu_{m-\frac{1}{2}})(v_i^-)^*v_j^-v_k^-$$
$$+hk_0^2\nu_{m+\frac{1}{2}}\sum_{i=0}^{3}f_i(\nu_{m+\frac{1}{2}})v_i^+ + hk_0^2\epsilon_{m+\frac{1}{2}}\sum_{i,j,k=0}^{3}g_{ijk}(\nu_{m+\frac{1}{2}})(v_i^+)^*v_j^+v_k^+ = 0.$$

- The scheme reduces to compact fourth order finite differences on smooth regions. For constant coefficients it is equivalent to

$$\frac{E_{m-1}-2E_m+E_{m+1}}{h^2}+k_0^2\nu\frac{E_{m-1}+10E_m+E_{m+1}}{12}=0.$$

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$$+\frac{h^2k_0^2}{24}\left(\epsilon_{m+\frac{1}{2}}\frac{|E_{m+1}|^2E_{m+1}-|E_m|^2E_m}{h}-\epsilon_{m-\frac{1}{2}}\frac{|E_m|^2E_m-|E_{m-1}|^2E_{m-1}}{h}\right)$$
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Two-way boundary conditions

- Obtained directly for the scheme, as in [Fibich & Tsynkov '01, '05].
- We exploit the fact that the schemes withstand the discontinuities, including those at the outer boundaries $z = 0$ and $z = Z_{\max}$.
- As $\nu_{m+\frac{1}{2}} \equiv 1$ and $\epsilon_{m+\frac{1}{2}} \equiv 0$ for $z_m \leq 0$, the schemes reduce to

$$L_1 E_{m-1} - 2L_0 E_m + L_1 E_{m+1} = 0, \quad m = 0, -1, \ldots,$$

where for the second order we have:

$$L_0 = (k_0 h)^{-2} - \frac{3}{8}, \qquad L_1 = (k_0 h)^{-2} + \frac{1}{8},$$

and for the fourth order we have:

$$L_0 = (k_0 h)^{-2} - \frac{1}{3} - \frac{3}{128}(k_0 h)^2, \qquad L_1 = (k_0 h)^{-2} + \frac{1}{6} + \frac{7}{384}(k_0 h)^2.$$

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Two-way boundary conditions (cont'd)

- General solution of the difference equation is $C_1 q^m + C_2 q^{-m}$, where

$$q = L_0/L_1 + i\sqrt{1-(L_0/L_1)^2} \text{ and } q^{-1} = L_0/L_1 - i\sqrt{1-(L_0/L_1)^2}$$

  are roots of the characteristic equation $L_1 q^2 - 2L_0 q + L_1 = 0$.
- $|q| = 1$, $q^{-1} = q^*$, $q^m \approx e^{ik_0 z} \equiv e^{ik_0 h(m-1)}$, $q^{-m} \approx e^{-ik_0 z} \equiv e^{-ik_0 h(m-1)}$.
- Instead of $E(z) = E^0_{\text{inc}} e^{ik_0 z} + Re^{-ik_0 z}$, $z \le 0$, we can then write:

$$E_m = E^0_{\text{inc}} q^{m-1} + Rq^{1-m}, \quad m = 1, 0, -1, \ldots$$

- Hence, we obtain the BC by expressing the ghost node value E_0:

$$E_0 = (q^{-1} - q)E^0_{\text{inc}} + qE_1.$$

- Similarly, at the right boundary $z = Z_{\max}$ we have:

$$E_{M+1} = qE_M.$$

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General comments on discretizations

- There are many coefficients f_i and g_{ijk}, but their computation can be efficiently automated.
- The schemes maintain their respective accuracy throughout the entire domain, including the points of material discontinuity.
- Outer boundaries are also points of discontinuity, and the BCs guarantee the same accuracy as that of the interior scheme.
- Only one ghost node is required for either second or fourth order.
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Iterative solver

- Previous solver (frozen nonlinearity + Born approximations) had convergence limitations far below the threshold of nonuniqueness.
- An alternative is **Newton's method.** However, the nonlinearity $|E|^2E$ is Frechét non-differentiable for complex-valued E.
- Fix: To recast the NLH as a system of two equations w.r.t. $\mathrm{Re}E$ and $\mathrm{Im}E$, and use real differentiation as opposed to Cauchy-Riemann.
- Major changes in the algorithm, including boundary conditions.
- The Jacobians are inverted by a direct method, which is perfectly feasible for 1D (and is also OK for moderate 2D).
- Choice of the initial guess:
  - Closed form exact solution — to test the convergence.
  - Continuation in ϵ: $\epsilon_0 \to \epsilon_1 \to \ldots \to \epsilon_n$, if the solution is not known.
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  - Continuation in ϵ: $\epsilon_0 \to \epsilon_1 \to \ldots \to \epsilon_n$, if the solution is not known.
- Altogether, Newton's method demonstrates robust convergence for high nonlinearities, way above the non-uniqueness threshold.

Iterative solver

- Previous solver (frozen nonlinearity + Born approximations) had convergence limitations far below the threshold of nonuniqueness.
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Local convergence of Newton's iterations

- The goal is to determine how the magnitude of nonlinearity, i.e., ϵ, affects Newton's convergence compared to the previous method.
- The initial guess is the trace of the exact solution on the grid.
- Original iterations converge only for $\epsilon < 0.167$.
- Newton's method converges all the way through $\epsilon = 3$, the highest value tried. In reality it would cause a breakdown of the material.
  - Convergence is rapid, 4–6 iterations reduce the residual by 10^{-9}–10^{-11}.
  - Convergence slows down and ceases near ϵ_c and ϵ_c', where $\left(\frac{dT}{d\epsilon}\right)^{-1} = 0$.
  - Convergence inside the switchback.
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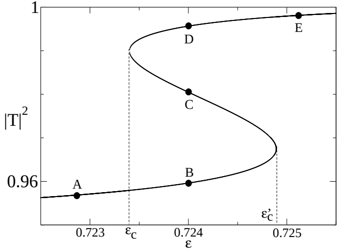

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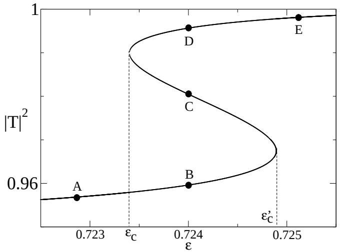

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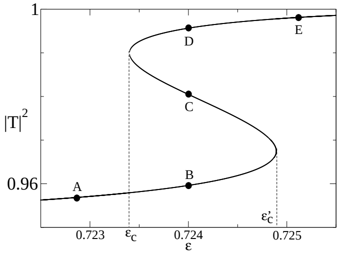

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Continuation approach

- The goal is to test Newton's convergence for the initial guesses that are not necessarily close to the solution.
- Example: for $E^{(0)} = e^{ik_0 z}$ there is no convergence if $\epsilon > 0.08$.
- Hence, for large ϵ we employ the continuation heuristics.
- The nonlinearity is increased in increments, $\epsilon_0 < \epsilon_1 < \ldots < \epsilon_n$, and the k-th solution is taken as the initial guess for $k+1$.
  - Allowable increments decrease as ϵ increases, but never reduce to zero.
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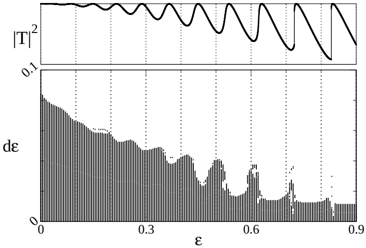

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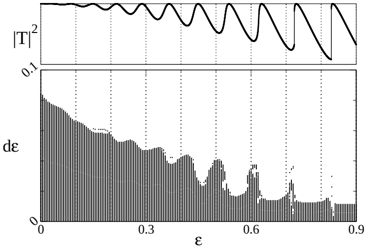

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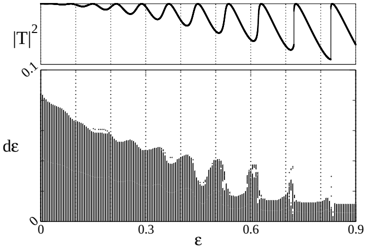

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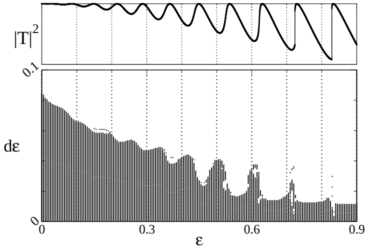

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Error for the homogeneous medium

- Discontinuities (jumps) are only at the boundaries $z = 0$, $z = Z_{\max}$.
- $\nu = 1.01^2$, $\epsilon = 0.01$ — small jumps and weak nonlinearity.
- $\nu = 1.3^2$, $\epsilon = 0.845$ — large jumps and strong nonlinearity.
- Comparison with the exact solution by Chen & Mills.
- $Z_{\max} = 10$, $k_0 = 8$. For $z \notin [0, Z_{\max}]$, $\nu = 1$ and $\epsilon = 0$.

ν	ϵ	$\tilde{h} \equiv hk_0$					
		$8 \cdot 10^{-1}$	$8 \cdot 10^{-1.5}$	$8 \cdot 10^{-2}$	$8 \cdot 10^{-2.5}$	$8 \cdot 10^{-3}$	Error($\tilde{h}$)
		Original central difference five-node $\mathcal{O}\left(h^4\right)$ scheme + Newton's solver					
1.01^2	0.01	0.187	$2.01 \cdot 10^{-3}$	$2.73 \cdot 10^{-5}$	$9.97 \cdot 10^{-7}$	$9.15 \cdot 10^{-8}$	$0.45 \cdot \tilde{h}^4 + 0.0024 \cdot \tilde{h}^2$
1.3^2	0.845	-	0.15	0.093	$5.40 \cdot 10^{-4}$	$5.38 \cdot 10^{-5}$	$24 \cdot \tilde{h}^4 + 0.84 \cdot \tilde{h}^2$
		Finite volume $\mathcal{O}\left(h^2\right)$ scheme + Newton's solver					
1.01^2	0.01	-	0.109	$1.09 \cdot 10^{-2}$	$1.09 \cdot 10^{-3}$	$1.09 \cdot 10^{-4}$	$1.71 \cdot \tilde{h}^2$
1.3^2	0.845	-	-	$2.36 \cdot 10^{-2}$	$2.01 \cdot 10^{-3}$	$1.98 \cdot 10^{-4}$	$3.72 \cdot \tilde{h}^2$
		Compact finite volume $\mathcal{O}\left(h^4\right)$ scheme + Newton's solver					
1.01^2	0.01	0.121	$1.29 \cdot 10^{-3}$	$1.28 \cdot 10^{-5}$	$1.28 \cdot 10^{-7}$	$1.33 \cdot 10^{-9}$	$0.314 \cdot \tilde{h}^4$
1.3^2	0.845	-	$8.16 \cdot 10^{-2}$	$9.12 \cdot 10^{-5}$	$9.13 \cdot 10^{-7}$	$9.16 \cdot 10^{-9}$	$2.23 \cdot \tilde{h}^4$

Error for the layered medium

- Additional discontinuity at the center of the domain:

$$\nu(z) = \begin{cases} 1.21 & z \in [0,5) \\ 1.69, & z \in (5,10] \end{cases}, \quad \epsilon(z) = \begin{cases} 0.1210 & z \in [0,5) \\ 0.5070, & z \in (5,10] \end{cases}.$$

- $Z_{\max} = 10$, $k_0 = 8$. For $z \notin [0, Z_{\max}]$, $\nu = 1$ and $\epsilon = 0$.
- Fourth order compact finite volume scheme + Newton's solver; comparison with the exact solution by Chen and Mills.

$\tilde{h} \equiv hk_0$					
$4 \cdot 10^{-1}$	$4 \cdot 10^{-1.5}$	$4 \cdot 10^{-2}$	$4 \cdot 10^{-2.5}$	$4 \cdot 10^{-3}$	Error$(\tilde{h})$
$3.70 \cdot 10^{-2}$	$3.72 \cdot 10^{-4}$	$3.69 \cdot 10^{-6}$	$3.69 \cdot 10^{-8}$	$3.93 \cdot 10^{-10}$	$1.42 \cdot \tilde{h}^4$

Summary

- Compact high order scheme for 1D NLH that maintains accuracy across discontinuities.
- Boundary conditions are built directly for the discretization. Compact schemes require only one ghost node.
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- Comparison with other work:
  - Our approach [Fibich & Tsynkov '01, '05] does not handle jumps and does not provide convergence for high nonlinearities.
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- Higher degrees of nonlinearity, e.g., quintic nonlinearity.
- Piecewise smooth material coefficients, as opposed to only piecewise constant.
- Discontinuities inside the cell, as opposed to only at the nodes.
- Absorption — complex-valued material coefficients.
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- Various numerical fixes aimed at speeding up the procedure.
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- Discontinuities inside the cell, as opposed to only at the nodes.
- Absorption — complex-valued material coefficients.
- Higher orders of accuracy (in fact, arbitrary) for the linear NLH.
- Various numerical fixes aimed at speeding up the procedure.
- **The extension of utmost interest — the multi-D NLH:**
  - Singular solutions or arrest of collapse?
  - Compact high order scheme can be constructed, but it will not be an automatic extension of 1D.
  - Key hurdle for Newton's method — inversion of the Jacobians. Will require preconditioned Krylov iterations.