

Mutual Fund Portfolio Choice in the Presence of Dynamic Flows¹

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February 12, 2004

¹We thank Pierre Collin Dufresne, Mike Galmeyer, Ming Huang, Lane Hughston, Monique Jeanblanc, Hong Liu, Patrice Poncet, Steve Shreve and Stan Zin as well as seminar participants at the Bachelier 2nd world congress, Berkeley, Carnegie-Mellon, CIRANO, Duke, HEC Montréal, Université Paris–1, Stanford, University of Texas at Austin, Washington University at St Louis and IBF at HEC Lausanne.

²Financial support from the United States National Science Foundation under grant DMS-9819950 and from IFM² is gratefully acknowledged.

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Abstract

We analyze the implications of the widely used fixed fraction of funds fees on a mutual fund manager's portfolio decisions. In our model, a log utility investor is allowed to dynamically allocate capital between an actively managed mutual fund and a locally riskless bond. The optimal fund portfolio is shown to be the one that maximizes the market value of the fees received, and is independent of the manager's utility function. The presence of dynamic flows induces "flow hedging" portfolio distortions on the part of the fund, even though the investor is myopic. Our model predicts a positive relationship between a fund's proportional fee rate and its volatility. This is a consequence of higher fee funds holding more extreme equity positions. While both the fund portfolio and the investor's trading strategy depend on the proportional fee rate, the welfares of the different agents do not. Implications related to the measured performance-fundflow relationship and its dependence on the fee rate are derived. Finally, we show that our results hold even if in addition to trading the fund and the bond the investor is allowed to directly trade some of the risky securities, but not all.

JEL Classification Numbers. G11, G12, G13, G23, D81.

Key words. Mutual Funds, Delegated Portfolio Management, Incomplete Markets, Continuous Time Stochastic Game, Backward Stochastic Differential Equations .

1 Introduction

The analysis of the impact of fee structures on a fund's portfolio choice decisions has previously typically been studied in a setting in which a manager receives an exogenously prespecified amount of money to manage for an exogenously prespecified time period. In the few models where investors do make allocation decisions, they are typically restricted to a one time decision, at the initial date, that determines the fraction of their initial wealth that they delegate to the management of the fund. Once this initial decision is made, investors are unable to delegate additional funds or to withdraw any part of the money under the fund's management, until the terminal date.

The important innovation of our paper is that we allow the investor to move money in and out of the fund dynamically. To this end, we study a dynamic continuous time economy with two agents: a *small investor* and a *fund manager*. The small investor implicitly faces high costs that preclude him from trading directly in the equity market. These implicit costs can be related, for example, to the fact that the opportunity cost for the investor of spending his time in stock trading related activities is high.¹ While he is precluded from holding equity directly, he can invest money in a mutual fund. Specifically, he is permitted to dynamically allocate his money between a mutual fund and a locally riskless bond, where we impose the natural restriction that the investor can not short the fund. The fund manager, on the other hand, is allowed to trade dynamically in both the stocks and the bond.

Our objective is to understand how enabling investors to dynamically rebalance between the fund and the bond impacts the trading strategy of the fund. We focus our analysis on the case of fraction of funds fees, whereby the manager receives a fee that is a fixed proportion of the assets under management. Our motivation for focusing such a fee structure stems from its widespread use by mutual funds. As of 1997, approximately 98% of all mutual funds were using a fraction of fund fees without any performance based incentives.

To focus on the impact of flows on a fund's portfolio choice decisions, while maintaining a tractable setup, we make the following simplifying assumptions. First, both agents have complete information about financial markets. Second, agents observe the actions of each other. Third, from the perspective of the fund manager markets are complete. Fourth, while only mild regularity conditions are imposed on the manager's utility function, the small investor is assumed to have a log utility function.

We further assume that while both agents observe the actions of each other, the fund manager is strategic where as the investor is not. Specifically, when the investor determines his portfolio he takes the fund strategy as given. On the other hand, when the fund manager selects the fund's trading strategy he takes into account what would be the investor's reaction to such a strategy.

When the investment opportunity set is stationary, keeping in mind that the investor's

¹This assumption is relaxed in Section 5 where we allow the investor direct access to a subset of the risky assets.

utility function is log, we obtain the intuitive result that the equilibrium fund portfolio is the portfolio that maximizes the after fees Sharpe ratio of the fund. In equilibrium, the equity component of the fund's portfolio is proportional to the fee rate, and the investor's holding of the fund is inversely proportional to the fee rate. An increase in the fee rate tilts the fund portfolio towards the risky assets, and at the same time decreases the investor's holdings in the fund. Furthermore, while both the fund's and the investor's portfolios depend on the fee rate, the investor's effective exposure to equity as well as both agents' expected utilities do not.

When the investment opportunity set is stochastic, it is no longer the case that the equilibrium fund portfolio is the one that maximizes the after fees Sharpe ratio of the fund. The presence of dynamic flows induces "flow hedging" on the part of the fund, even though the investor is myopic. In selecting a fund portfolio the manager takes account of the fact that the fraction of wealth delegated to the fund's management by a myopic log investor is determined by the instantaneous Sharpe ratio provided by such an investment. As a result, in states of the world where the opportunity set is good (bad) and the Sharpe ratio is high (low) the manager will receive a larger fraction of the investor's wealth as management fees. This leads the manager to distort the fund's equity portfolio so that the future opportunity set (i.e., the Sharpe ratio) is correlated with the investor's wealth process. Interestingly, the distortions within the equity component do not depend on the fee rate. However, even with a stochastic opportunity set the investor's effective equity exposure and the expected utilities of the two agents are independent of the fee rate.

We also use our model to investigate a potential contemporaneous relationship between flows and fund returns. For low fee rates our model predicts that high fund returns will be accompanied by inflows whereas low fund returns will be accompanied by outflows. The analysis in the empirical literature on the relationship between flows and lagged fund returns typically does not control for a contemporaneous relationship (see for example Sirri and Tufano (1998)); thus, potentially biasing the results. In addition, we derive empirical predictions related to the relationship between fee rates and flows, holding fund returns fixed.

A different aspect of dynamic fund flows is analyzed by Berk and Green (2002) who demonstrate that a combination of unknown fund manager ability and decreasing returns to scale can generate the following two empirical facts. First, a positive convex relationship between lagged fund returns and flows into the fund. Second, non predictable fund returns even after conditioning on past performance.

Since the focus of the present paper is to analyze the impact of dynamic flows on a mutual fund manager's portfolio choice decisions, we take the fee structure as given. We are not taking a stance on whether a fraction of funds fees is the form of the optimal compensation contract in our setting, but instead rely on its widespread use by mutual funds as the motivation for our analysis. Papers that analyze these optimal contracting issues include Starks (1987), Roll (1992), Heinkel and Stoughton (1994), Admati and Pfleiderer (1997), Lynch and Musto (1997), Das and Sundaram (1998a,b), Dybvig, Farnsworth and Carpenter

(1999) and Ou-Yang (2002).

While fraction of funds fees is by far the predominant compensation contract in the mutual fund industry some funds have a symmetric performance component in their compensation contract (a fulcrum fee). Furthermore, both pension funds and hedge funds use asymmetric performance contracts in which the manager shares in the upside, but does not get penalized on the downside. Grinblatt and Titman (1989), Chen and Pennachi (1999), and Carpenter (2000) have studied the optimal portfolio strategy of a manager receiving convex performance fees in a setting in which the manager receives an exogenous amount of money to manage at the initial date. An analysis of the equilibrium asset pricing implications, as well as the fund portfolio choice decisions, of both fulcrum fees and asymmetric performance fees is conducted in Cuoco and Kaniel (2001). In Cuoco and Kaniel (2001) both fund managers and unconstrained investors trade directly in the equity markets, but the investors who use the funds services make allocation decisions only at the initial date. High water mark fees, used in the hedge fund industry, are discussed in Goetzmann, Ingersoll and Ross (2004).

In solving our model we rely on tools that were developed for the study of utility maximization problems under incomplete markets as well as on the theory of Backward Stochastic Differential Equations and its links to the study of optimal control problems.² For details on the study of utility maximization problems in incomplete or constrained financial markets, see Karatzas et al. (1991), He & Pearson (1991) and Cvitanić & Karatzas (1992) among others. For a comprehensive survey of backward stochastic differential equations and some of their applications in finance, see El Karoui, Peng and Quenez (1997). A brief introduction to the theory of such equations as well as a review of the results needed in the paper is given in Appendix B.

The remainder of the paper is organized as follows. The economic setup is described in Section 2. In Section 3 we solve the investor's optimization problem and derive the optimal fund portfolio as well as the manager's optimal trading strategy. Section 4 describes the equilibrium trading strategies and discusses the models implications, including implications related to the performance-flow relationship. Section 5 demonstrates that the main results hold even if one relaxes the assumption that the investor can get exposure to equity only through holding the fund. Section 6 concludes. Appendix A contains all the proofs and Appendix B contains a short review of the necessary results from the theory of Backward Stochastic Differential Equations.

2 The Economy

We consider a continuous time economy on the finite time span $[0, T]$. The uncertainty is represented by a probability space $(\Omega, \mathbb{F}, \mathcal{F}, P)$ on which is defined an n -dimensional

²Although the case of constant market coefficients can be solved explicitly without using backward stochastic differential equations, it is important to note that the use of such equations allows us to solve the problem completely even in the case of non Markovian dynamics for the traded securities.

standard Brownian motion B . The filtration $\mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T}$ is the usual augmentation of the filtration generated by the Brownian motion and we let $\mathcal{F} := \mathcal{F}_T$ so that the true state of nature is solely determined by the path of the Brownian motion up to the terminal time. In the sequel, all processes are assumed to be adapted to the filtration \mathbb{F} and all statements involving random quantities are understood to hold either almost surely or almost everywhere depending on the context. Furthermore, we shall assume all the processes introduced to be well defined, without explicitly stating the regularity conditions ensuring this.

2.1 Securities and Mutual Fund Dynamics

2.1.1 Securities

There is a single perishable good (the numéraire) in units of which all quantities are expressed. The financial market consists of $n + 1$ long lived securities. The first security is a locally riskless bond, whose price process S_0 satisfies

$$dS_t^0 = r_t S_t^0 dt, \quad S_0^0 = 1$$

for some *interest rate* process r . The remaining securities are risky. They are referred to as the *stocks* and their price process $S := (S^i)_{i=1}^n$ satisfies

$$dS_t = \text{diag}[S_t] \left(a_t dt + \sigma_t dB_t \right), \quad S_0 \in (0, \infty)^n$$

for some vector valued *drift* process a and some matrix valued *volatility* process σ . The processes r , a and σ are collectively referred to as the model coefficients and will be assumed to satisfy the following

Assumption 1 *The model coefficients are uniformly bounded processes and furthermore, the volatility process admits a uniformly bounded inverse.*

The assumption imposed on the model coefficients imply that the relative risk premium process $\xi := \sigma^{-1}[a - r\mathbf{1}]$, where $\mathbf{1}$ denotes the n -dimensional unit vector, is uniformly bounded. The exponential local martingale

$$M_t := \exp \left[- \int_0^t \xi_\tau^* dB_\tau - \frac{1}{2} \int_0^t \|\xi_\tau\|^2 d\tau \right]$$

where $\|\cdot\|$ denotes the Euclidean norm, is thus a strictly positive and uniformly integrable martingale. As a result, the formula $Q(A) := E[1_A M_T]$ defines a probability measure which is equivalent to the original probability measure and under which the process

$$Z_t := B_t + \int_0^t \xi_\tau d\tau \tag{1}$$

is an n -dimensional standard Brownian motion by Girsanov theorem. Under this new probability measure, the dynamics of the stock price are given by

$$dS_t = \text{diag}[S_t] \left(r_t \mathbf{1} dt + \sigma_t dZ_t \right).$$

The discounted stock price process S/S^0 is thus a martingale under Q and is referred to as the *risk neutral* or *equivalent martingale measure*.

2.1.2 Mutual Fund Dynamics

Trading takes place continuously and there are no market frictions. A mutual fund trading strategy is a vector valued process $\pi = (\pi^k)_{k=1}^n$ specifying the portfolio weight in each of the stocks. Given a fund trading strategy, the return on investments in the fund at time t evolves according to

$$\begin{aligned} \frac{dF_t}{F_t} &= \left[1 - \pi_t^* \mathbf{1}\right] \frac{dS_t^0}{S_t^0} + \sum_{k=1}^n \pi_t^k \frac{dS_t^k}{S_t^k} - \gamma dt \\ &= (r_t - \gamma)dt + \pi_t^* \sigma_t \left[dB_t + \xi_t dt \right] = (r_t - \gamma)dt + \pi_t^* \sigma_t dZ_t \end{aligned} \quad (2)$$

where the constant $\gamma > 0$ represents the rate at which management fees are withdrawn from the asset under management.³

Throughout the rest of the paper we use the notation $F = F^\pi$ to emphasize the dependence on the portfolio weight process chosen by the fund manager, and denote by Π_f the set of fund portfolio processes with the property that the associated solution to (2) is non negative.

2.2 Agents

2.2.1 The Investor

The small investor, which we denote by i , is allowed to trade freely in the riskless security but can only access the market for risky securities through the fund. The inability of the investor to trade stocks directly should be viewed as a reduced form representing the fact that it is more costly for the investor to trade efficiently stocks than it is for the fund. This could be due, for example, to the fact that given that the investor is small it will be costly for him to hold a portfolio that consists of many stocks. Furthermore, it is implicitly assumed that the opportunity cost for the investor of spending his time in stock trading related activities is high. In the last section of the paper we relax this assumption by allowing the investor to trade directly in a subset of the stocks.

We denote by W_0^i the investor's initial capital and by ϕ_t the amount he holds in the mutual fund at time t . Since investors can not short actively managed funds, we impose the constraint

$$\phi_t \geq 0, \quad 0 \leq t \leq T.$$

³In practice, both management fees and distribution (12b-1) fees are indeed deducted continually from assets under management. Some funds also impose either front end or back end load fees; which we ignore in our analysis.

Given an admissible fund portfolio $\pi \in \Pi_f$, and under the usual self financing condition, the investor's wealth process evolves according to

$$\begin{aligned} dW_t^i &= \left[W_t^i - \phi_t \right] \frac{dS_t^0}{S_t^0} + \phi_t \frac{dF_t^\pi}{F_t^\pi} \\ &= r_t W_t^i dt + \phi_t \pi_t^* \sigma_t \left[dB_t + \xi_t^\pi dt \right] \end{aligned} \quad (3)$$

with initial condition W_0^i where the process ξ_t^π is the risk premium associated with an investment in the stocks through the mutual fund, that is

$$\xi_t^\pi := \xi_t - \gamma \frac{\sigma_t^* \pi_t}{\|\sigma_t^* \pi_t\|^2}. \quad (4)$$

For the portfolio management services provided by the fund, the investor is charged a fee that is proportional to the amount he currently holds in the fund. Specifically, as shown in Equation (2), at each time t the investor is charged a flow of $\gamma \phi_t$ where $\gamma \in \mathbb{R}_+$ is a given *management fee rate*. The cumulative fee process, which in turn determines the manager's cumulative income process, is thus given by

$$\Phi_t := \int_0^t \gamma \phi_\tau d\tau. \quad (5)$$

Since the investor's sole resource is his initial capital, he will be able to avoid bankruptcy only if his wealth remains positive. In what follows, we denote by Θ_i^π the set of non negative processes ϕ with the properties that

$$E_Q[\Phi_T] = E_Q \left[\int_0^T \gamma \phi_\tau d\tau \right] < \infty, \quad (6)$$

where $E_Q[\cdot]$ denotes the expectation operator under the risk neutral probability measure, and that given π the solution to (3) is a positive process.

The investor is assumed to have logarithmic preferences over terminal wealth. Given a fund portfolio process $\pi \in \Pi_f$, his objective is thus to select a trading strategy $\phi \in \Theta_i^\pi$ so as to maximize the expected utility

$$E[\log(W_T^i)].$$

The fact that the fund value is driven by n independent Brownian motions and that the investor is not allowed to short the fund imply that the investor effectively faces an incomplete market. The assumption of log utility is therefore critical for tractability, since this utility function is the only one allowing for a closed form solution in our possibly non Markovian framework.⁴

⁴Under the assumption of constant market coefficients, the model could also be solved with a constant relative risk aversion utility function for the investor. However, the results in this case are qualitatively similar to those with a logarithmic utility function.

2.2.2 The Fund Manager

The fund manager, which we denote by m , determines both the fund's trading strategy and a trading strategy in his own account. He is endowed with a non negative initial capital W_0^m and receives income from his management activities according to the cumulative fee process generated by the investor's holdings of the mutual fund.

The fact that we allow the manager to trade stocks directly in his own account, while restricting the ability of the small investor to do so, is motivated by the fact that the manager is exposed to equity markets through his job. Thus, his implicit costs of participating in stock trading activities are far lower than those of the investor's. In Section 5 we allow the small investor direct access to some of the assets; thereby decreasing the wedge between the implicit costs to the investor and to the manager.⁵

Let Φ be given by (5) for some $\phi \in \Theta_i^\pi$ and assume that the manager trades in his own account according to some trading strategy η . Under the usual self financing condition, the manager's wealth process then satisfies the dynamic budget constraint

$$\begin{aligned} dW_t^m &= \left[W_t^m - \eta_t^* \mathbf{1} \right] \frac{dS_t^0}{S_t^0} + \sum_{k=1}^n \eta_t^k \frac{dS_t^k}{S_t^k} + d\Phi_t \\ &= r_t W_t^m dt + \eta_t^* \sigma_t \left[dB_t + \xi_t dt \right] + \gamma \phi_t dt. \end{aligned} \quad (7)$$

In what follows, we say that a trading strategy is admissible for the manager and we write $\eta \in \Theta_m$ if the associated wealth process is such that $W^m \geq -\Gamma$ for some non negative and Q -integrable random variable.

Given a cumulative income process as in (5), the manager's objective is to find an admissible trading strategy which maximizes the expected utility

$$E [U(W_T^m)].$$

In what follows, the manager's utility function $U : (0, \infty) \rightarrow \mathbb{R}$ is assumed to be strictly increasing, strictly concave, continuously differentiable and to satisfy the Inada conditions at both zero and infinity.

2.3 Equilibrium

A Nash equilibrium for our delegated portfolio management game is defined by a triple of dynamic trading strategies $(\hat{\pi}, \hat{\eta}, \hat{\phi}) \in \Pi_f \times \Theta_m \times \Theta_i^{\hat{\pi}}$ with the following properties

- a) The trading strategy $\hat{\phi}$ is optimal for the investor given the fund portfolio process $\hat{\pi}$ chosen by the manager, that is:

$$\hat{\phi} \in \arg \max_{\phi \in \Theta_i^{\hat{\pi}}} E [\log(W_T^i)].$$

⁵While information is symmetric in our model, part of the wedge in the costs of directly trading equity between the manager and the investor can be intuitively viewed as being driven by differences in information costs.

- b) The combination of the fund portfolio process $\hat{\pi}$ and the trading strategy $\eta^{\hat{\pi}}$ is optimal for the fund manager given the cumulative income process generated by the investor's trading strategy, that is:

$$(\hat{\pi}, \eta^{\hat{\pi}}) \in \arg \max_{(\pi, \eta) \in \Pi_f \times \Theta_m} E[U(W_T^m)] \quad \text{given } \Phi_t = \Phi_t^{\hat{\pi}} := \int_0^t \gamma \phi_{\tau}^{\hat{\pi}} d\tau$$

The focus of our analysis is on how dynamic flows impact a fund's portfolio choice decisions. As such, in this paper we view the investor as a *small* agent who does not realize that his investment decisions (i.e., the amount of money he delegates to the fund to manage at each point in time) may impact the fund's portfolio choice decision. The fund manager on the other hand is strategic. Specifically, we assume that in making his investment decisions, the manager takes into account the impact of the fund portfolio on the investor's demand for the mutual fund.

As a result, one can interpret our model as a stochastic differential game between the manager and the investor, where the manager's strategic advantage resembles that of a Stackelberg leader. This suggests (see Bagchi (1984)) that the dominating, in the sense of being both individually and collectively stable, equilibrium can be computed through the following sequential procedure.

1. The first step consists in solving the investor's maximization problem given an arbitrary fund portfolio process $\pi \in \Pi_f$ to obtain the trading strategy $\phi^{\pi} \in \Theta_i^{\pi}$ which is the investor's best response to π . For each portfolio process π this optimal strategy induces via (5) a cumulative fee process Φ^{π} that is payed to the manager.
2. The second step consists in solving the manager's maximization problem given the cumulative fee process Φ^{π} in order to obtain his optimal trading strategy $\eta^{\pi} \in \Theta_m$ as well as his value function

$$V_m^{\pi} := \left\{ \max_{\eta \in \Theta_m} E[U(W_T^m)] \text{ given } \Phi = \Phi^{\pi} \right\}. \quad (8)$$

3. The third step makes the connection between the two individual optimization problems and determines, from the manager's perspective, the optimal fund portfolio process:

$$\hat{\pi} \in \arg \max_{\pi \in \Pi_f} V_m^{\pi} = \arg \max_{\pi \in \Pi_f} \left\{ \max_{\eta \in \Theta_m} E[U(W_T^m)] \text{ given } \Phi = \Phi^{\pi} \right\}. \quad (9)$$

Re-injecting this fund portfolio process in the first two steps we obtain a pair of optimal trading strategies. By construction the triple $(\hat{\pi}, \eta^{\hat{\pi}}, \phi^{\hat{\pi}})$ is a Nash equilibrium of the game, and since the definition of the optimal fund portfolio process implies that the corresponding allocations are Pareto optimal, it has to be that this non cooperative equilibrium is dominating.

3 Optimal Trading Strategies

In solving for the optimal trading strategies of the two agents, we follow the sequential procedure described in the previous section. The first step is carried in Section 3.1, and the second and third in Section 3.2.

3.1 The Investor's Problem

In this section we assume that the fund portfolio is a fixed process $\pi \in \Pi_f$, and derive the small investor's best response trading strategy.

Given a fund portfolio π and the dynamics (2), the net of fees instantaneous Sharpe ratio of the fund is given by

$$\vartheta_t^\pi := \frac{[r_t - \gamma + \pi_t^* \sigma_t \xi_t] - r_t}{\|\sigma_t^* \pi_t\|^2} = \frac{\pi_t^* \sigma_t \xi_t - \gamma}{\|\sigma_t^* \pi_t\|^2} \quad (10)$$

Combining the fact that the investor's utility function is log with the fact that he can not short the fund, it is natural to conjecture that at any point in time the investor will hold the fund only if it has a positive net of fees Sharpe ratio, and that in those cases the proportion of his wealth that will be invested in the fund would equal that Sharpe ratio. The following proposition shows that this is indeed the case.

Proposition 1 *Let $\pi \in \Pi_f$ denote an arbitrary fund portfolio process and define a non negative process by setting*

$$\phi_t^\pi := [\vartheta_t^\pi]^+ W_t^{i\pi} = \max[0, \vartheta_t^\pi] W_t^{i\pi} \quad (11)$$

where $W^{i\pi}$ is the corresponding solution to (3). Then ϕ^π belongs to Θ_i^π and constitutes the investor's best response trading strategy.

3.2 The Manager's Problem

We start by assuming that the fund portfolio is a fixed process and solve the manager's problem (8), given the investor's best response trading strategy. Then, we complete our sequential optimization procedure by determining the optimal fund portfolio process.

3.2.1 Optimal Trading Strategy

Proposition 1 shows that for a given fund portfolio process, the investor's best response trading strategy generates the cumulative fee process Φ^π defined by

$$\Phi_t^\pi := \int_0^t \gamma \phi_\tau^\pi d\tau = \int_0^t \gamma [\vartheta_\tau^\pi]^+ W_\tau^{i\pi} d\tau \quad (12)$$

where $W^{i\pi}$ is the investor's optimal wealth process. Given that the stock price volatility matrix is invertible, the manager faces a complete financial market. In such a setting, the

solution to (8) is well-known (see Karatzas and Shreve (1999)). In order to describe it, we start by introducing the market value process

$$v_t^\pi := E_Q \left[\int_t^T \frac{d\Phi_\tau^\pi}{S_\tau^0/S_t^0} \middle| \mathcal{F}_t \right]$$

of the cumulative fee process generated by the investor's best response strategy. Standard results from the theory of complete markets then imply that the manager's problem amounts to the maximization of his expected utility over the set of non negative random variables which satisfy the static budget constraint

$$E_Q \left[\frac{W_T^m}{S_T^0} \right] \leq v_0^\pi + W_0^m \quad (13)$$

Let now I denote the strictly positive and strictly decreasing inverse of the manager's marginal utility function. The manager's utility function being strictly concave, the optimal terminal wealth for problem (8) can be written as

$$W_T^{m\pi} := I \left(y_\pi \frac{M_T}{S_T^0} \right)$$

where y_π is a strictly positive Lagrange multiplier chosen such that the static budget constraint (13) holds as an equality:

$$E_Q \left[\frac{1}{S_T^0} I \left(y_\pi \frac{M_T}{S_T^0} \right) \right] = v_0^\pi + W_0^m. \quad (14)$$

The following proposition summarizes the above discussion and makes use of the Brownian martingale representation theorem (Karatzas & Shreve (1988, Theorem 3.4.2 p.170)) in order to obtain an explicit characterization of the manager's optimal trading strategy.

Proposition 2 *Let $\pi \in \Pi_f$ denote an arbitrary fund portfolio process, assume that (14) admits a strictly positive solution y_π and define*

$$\frac{W_t^{m\pi}}{S_t^0} := E_Q \left[\frac{W_T^{m\pi}}{S_T^0} \middle| \mathcal{F}_t \right] - \frac{v_t^\pi}{S_t^0} = W_0^m + \int_0^t \left(h_\tau^* dZ_\tau + \frac{d\Phi_\tau^\pi}{S_\tau^0} \right). \quad (15)$$

Then the process $W^{m\pi}$ satisfies the manager's dynamic budget constraint (7) with the admissible trading strategy $\eta^\pi := [\sigma^]^{-1} S^0 h \in \Theta_m$ and its terminal value is optimal for the manager's utility maximization problem.*

3.2.2 Optimal Fund Portfolio

In order to determine the equilibrium trading strategies, we now need to find a fund portfolio process $\hat{\pi} \in \Pi_f$ that maximizes the manager's value function

$$V_m^\pi := E \left[U(W_T^{m\pi}) \right] = E \left[U \circ I \left(y_\pi \frac{M_T}{S_T^0} \right) \right] \quad (16)$$

Observing that the right hand side of the above equation depends on the fund portfolio only through the market value of the corresponding fees and using the decrease of the map $v_0^\pi \mapsto y_\pi$ in conjunction with that of the function $U \circ I$, it is easily seen that the choice of such a process is *independent* from the manager's utility function and amounts to the maximization of the initial market value of the fees. More precisely, the optimal fund portfolio process is the solution to the stochastic control problem with value function

$$\widehat{v}_0 := \sup_{\pi \in \Pi_f} v_0^\pi = \sup_{\pi \in \Pi_f} E_Q \left[\int_0^T \frac{d\Phi_\tau^\pi}{S_\tau^0} \right] \quad (17)$$

where the non decreasing process Φ^π is defined as in (12). Our first result in this section establishes a technical property which will be useful in what follows.

Lemma 1 *Fix an arbitrary fund portfolio process $\pi \in \Pi_f$ and let $W^{i\pi}$ denote the corresponding best response wealth process. Then the process defined by*

$$M_t^\pi := \exp \left(\int_0^t \gamma [\vartheta_\tau^\pi]^+ d\tau \right) \left[\frac{W_t^{i\pi}}{S_t^0 W_0^i} \right] \quad (18)$$

with the process ϑ^π as in (10), is a strictly positive and uniformly integrable martingale under the risk neutral probability measure.

Remark 1 Let $\pi \in \Pi_f$ denote an arbitrary fund portfolio process. The integration of the stochastic differential equation (3) verified by the investor's best response wealth process shows that we have

$$\int_0^t \frac{d\Phi_\tau^\pi}{S_\tau^0} = W_0^i - \frac{W_t^{i\pi}}{S_t^0} + \int_0^t \left(\frac{\phi_\tau^\pi \pi_\tau^* \sigma_\tau}{S_\tau^0} \right) dZ_\tau$$

Taking expectations under the risk neutral probability measure on both sides and using the fact that, thanks to Lemma 1, the local martingale on the far right is a uniformly integrable martingale, we obtain that the optimal fund portfolio problem (17) can be written as

$$\widehat{v}_0 = \sup_{\pi \in \Pi_f} \left\{ W_0^i - E_Q \left[\frac{W_T^{i\pi}}{S_T^0} \right] \right\} \quad (19)$$

Given the results in the previous sections we can interpret the objective function in (19) as the difference between the investor's initial wealth and the minimal amount required by an agent who has complete access to financial markets in order to generate the investor's optimal terminal wealth $W_T^{i\pi}$. The discrepancy between these two amounts generates the initial market value of the cumulative fee process to be received by the manager. Thus, we are prompted to conclude that the optimal fund portfolio is that which "maximizes the incompleteness" of the financial market to which the investor has access to.

Combining the result of Lemma 1 with Girsanov theorem, we have that for an arbitrary $\pi \in \Pi_f$ the formula $Q^\pi(A) := E_Q[1_A M_T^\pi]$ defines a probability measure which is equivalent to the risk neutral probability measure and under which the process

$$Z_t^\pi := Z_t + \int_0^t [\vartheta_\tau^\pi]^+ \sigma_\tau^* \pi_\tau d\tau$$

is an n -dimensional standard Brownian motion. In conjunction with the non negativity of the investor's initial wealth, the definition of this new probability measure shows that (19) can be written as

$$\widehat{v}_0 = W_0^i \left\{ 1 - \inf_{\pi \in \Pi_f} E_{Q^\pi} \left[e^{-\int_0^T \gamma [\vartheta_\tau^\pi]^+ d\tau} \right] \right\}. \quad (20)$$

The approach we take to solve this problem relies on the interpretation of its objective function as solution to a backward stochastic differential equation and on the comparison theorems for such equations (see the Appendix for a brief introduction to this theory).

Fix an arbitrary fund portfolio process, let

$$f(t, \pi_t) = [\vartheta_t^\pi]^+ = \frac{[\pi_t^* \sigma_t \xi_t - \gamma]^+}{\|\sigma_t^* \pi_t\|^2}$$

denote the positive part of the corresponding net of fees Sharpe ratio and consider the backward stochastic differential equation (BSDE in short)

$$-dX_t^\pi = f(t, \pi_t) (\pi_t^* \sigma_t Y_t - \gamma X_t^\pi) dt - Y_t^* dZ_t, \quad X_T^\pi = 1 \quad (21)$$

for the *pair* of processes (X^π, Y) . The map f being bounded (see the proof of Lemma 1 for details), it follows from Theorem B.1 in the Appendix that the unique square integrable solution to this equation is given by

$$X_t^\pi = E_{Q^\pi} \left[e^{-\int_t^T \gamma [\vartheta_\tau^\pi]^+ d\tau} \middle| \mathcal{F}_t \right]$$

and by the integrand in the representation of the associated Q^π -martingale. Comparing the above expression with (20) it is now easily seen that the value function of our optimization problem can be written as $W_0^i(1 - X_0)$ where

$$X_t := \operatorname{ess\,inf}_{\pi \in \Pi_f} X_t^\pi = \operatorname{ess\,inf}_{\pi \in \Pi_f} E_{Q^\pi} \left[e^{-\int_t^T \gamma [\vartheta_\tau^\pi]^+ d\tau} \middle| \mathcal{F}_t \right] \quad (22)$$

Given that the terminal condition in (21) is constant, the comparison theorems stated in the Appendix suggest that in order to solve for the optimal fund portfolio one should go through the following steps:

1. Perform the pointwise minimization of the driver

$$g(t, \pi, x, y) := f(t, \pi) (\pi^* \sigma_t y - \gamma x)$$

of the BSDE (21) over the set of admissible fund portfolios to obtain a candidate optimal control as a function of (t, x, y) and the state.

2. Plug this candidate optimal control back into the driver and show that the resulting BSDE admits a solution whose trajectory coincides with the process of (22).

A straightforward, albeit tedious, computation reveals that the unique minimizer of the mapping defined by (23) is the process given by

$$\widehat{\pi}(t, x, y) := 1_{\{\|y/x\| \leq \|\xi_t\|\}} [\sigma_t^*]^{-1} \frac{2\gamma(\xi_t - \frac{y}{x})}{\|\xi_t\|^2 - \|\frac{y}{x}\|^2}.$$

Plugging this candidate optimal fund portfolio process back into the definition of g our task now consists in establishing that the BSDE associated, under the risk neutral probability measure, with the driver

$$\begin{aligned} \widehat{g}[t, x, y] &:= \min_{\pi \in \mathbb{R}^n} g[t, \pi, x, y] = g[t, \widehat{\pi}(t, x, y), x, y] \\ &= -1_{\{\|y/x\| \leq \|\xi_t\|\}} \frac{x}{4} \|\xi(t) - \frac{y}{x}\|^2 \end{aligned}$$

and the terminal condition 1 admits a solution which coincides with the process of essential infima defined in (22). As in the traditional dynamic programming approach to Markovian stochastic control problems, where the main difficulty is to prove the existence of a classical solution of the associated Hamilton-Jacobi-Bellman equation, it is the proof of the existence of a solution to this *nonlinear* BSDE which constitutes the main difficulty of our approach. In particular, because the driver of the BSDE

$$-dX_t = \widehat{g}(t, X_t, Y_t)dt - Y_t^* dZ_t, \quad X(T) = 1 \quad (23)$$

has quadratic growth in Y , the standard existence results do not apply and it is well-known that even in the forward case such equations do not necessarily admit strong solutions. Nevertheless, exploiting the particular structure of the driver under consideration and using the results of Kobylanski (1997) and Lepeltier and San Martin (1998), we are able to establish the existence of a *maximal* solution to (23) and this will prove sufficient for our purposes.

Lemma 2 *The set of square integrable solutions to the BSDE (23) is non empty and contains an element whose trajectory is strictly positive and dominates that of any other solution.*

By combining the above result with the definition of the candidate optimal control and the refined comparison theorem of Lepeltier & San Martin (1998), we now obtain a complete solution to the fund portfolio selection problem.

Proposition 3 *Let $\beta := X^{-1}Y$ where (X, Y) denotes the maximal solution to the BSDE (23) and define a vector valued process by setting*

$$\widehat{\pi}_t := \widehat{\pi}(t, X_t, Y_t) = 1_{\{\|\beta_t\| \leq \|\xi_t\|\}} [\sigma_t^*]^{-1} \frac{2\gamma(\xi_t - \beta_t)}{\|\xi_t\|^2 - \|\beta_t\|^2}. \quad (24)$$

Then the process $\hat{\pi}$ belongs to the set Π_f of admissible fund portfolio processes and attains the infimum in the stochastic control problem (20).

Before turning to the determination of the agents' equilibrium value functions and trading strategies, let us first comment on the result of Proposition 3. Proposition 1 suggests that, since the investor's demand for the fund (and hence the manager's cumulative fee process) is increasing in the fund's net of fees Sharpe ratio, the optimal fund portfolio process should be that which maximizes the fund's Sharpe ratio in every state. A straightforward computation shows that the unique maximizer of the fund's Sharpe ratio is the process defined by

$$\tilde{\pi}_t := \arg \max_{\pi \in \mathbb{R}^n} f(t, \pi) = [\sigma_t^*]^{-1} \frac{2\gamma \xi_t}{\|\xi_t\|^2}. \quad (25)$$

Comparing this with the optimal fund portfolio process defined by (24) we see that the intuition according to which $\tilde{\pi}$ should be the optimal fund portfolio process is in general false. As we now demonstrate, this intuition is valid if the manager's investment opportunity set is stationary in the sense that the risk premium process ξ is deterministic.

Corollary 1 *Assume that the risk premium process ξ is a deterministic and bounded function. Then the maximal solution to the BSDE (23) is given by*

$$X_t := \exp \left[- \int_t^T \frac{\|\xi_\tau\|^2}{4} d\tau \right] \quad (26)$$

and $Y := (0, \dots, 0)^* \in \mathbb{R}^n$. In this case, the optimal fund portfolio process is also deterministic and coincides with the bounded process $\tilde{\pi}$ defined by (25).⁶

With a deterministic risk premium process ξ , combining (24) and (25) with Proposition 1 one might be tempted to also conclude that the optimal fund portfolio is the one that maximizes the investors welfare. This is of course false, as it is straightforward to show that if the investor was to determine the fund's portfolio, while still paying the applicable management fees, he would choose it to maximize $\|\sigma_t^* \pi_t\| [\vartheta_t^\pi]^+$ and not $[\vartheta_t^\pi]^+$.

4 Equilibrium and Fund Flows

4.1 Equilibrium

We now gather the previous results to obtain a complete description of the equilibrium. We start with the general case in which the market coefficients are arbitrary processes satisfying

⁶The result of this corollary can be obtained without using BSDEs. Indeed, using the definition of the process X^π in conjunction with the fact that the process $\tilde{\pi}$ is now deterministic we obtain that the inequality

$$\exp \left[- \int_t^T \frac{\|\xi_\tau\|^2}{4} d\tau \right] = E_{Q^\pi} \left[\exp \left(- \int_t^T \gamma [\vartheta_\tau^{\tilde{\pi}}]^+ d\tau \right) \right] \leq X_t^\pi$$

holds for every admissible fund portfolio process and the result follows from observing that the deterministic process $\tilde{\pi}$ belongs to the set Π_f .

the conditions of Assumption 1 and then proceed with the special case in which they are deterministic.

Theorem 1 *Let (X, Y) denote the maximal solution to the backward stochastic differential equation given by (23) and define $\beta := X^{-1}Y$. Then the equilibrium fund portfolio and the investor's equilibrium trading strategy are given by*

$$\hat{\pi}_t = 1_{\{\mathcal{B}\}} [\sigma_t^*]^{-1} \frac{2\gamma(\xi_t - \beta_t)}{\|\xi_t\|^2 - \|\beta_t\|^2}, \quad (27)$$

$$\phi_t^{\hat{\pi}} = 1_{\{\mathcal{B}\}} \left(\|\xi_t\|^2 - \|\beta_t\|^2 \right) \left[\frac{W_t^{i\hat{\pi}}}{4\gamma} \right] \quad (28)$$

where $\mathcal{B} \in \mathcal{F}_T$ denotes the set on which we have $\|\beta\| \leq \|\xi\|$. The equilibrium expected utilities of each of the two agents are given by

$$V_m := E \left[U \circ I \left(y_{\hat{\pi}} \frac{M_T}{S_T^0} \right) \right] \quad (29)$$

$$V_i := E \left[\log (W_0^i S_T^0) \right] + E \left[\int_0^T \frac{1_{\{\mathcal{B}\}} \|\xi_\tau - \beta_\tau\|^2}{8} d\tau \right] \quad (30)$$

where $y_{\hat{\pi}}$ solves (14) with $v_0^{\hat{\pi}} = W_0^i(1 - X_0)$. Finally, the manager's equilibrium trading strategy on his own account is given by $\eta^{\hat{\pi}} = [\sigma^*]^{-1} S^0 h$ where the process h is the integrand in the representation of the martingale defined by (15).

Equation (27) shows that in equilibrium the equity component of the fund is proportional to the fee rate. Thus, our model predicts that *ceteris paribus* funds with high proportional management fees will invest more in equities relative to funds with low proportional management fees. In particular, this implies a positive relationship between a fund's fee rate and its volatility.

Equation (28) shows that, as can be expected, the higher the fee rate the lower the fraction of the investor's wealth that is invested in the fund. The fee being charged on assets under management, it applies to both the equity and the bond components of the fund. Obviously, holding the bond through the fund is more expensive for the investor than holding it directly. To mitigate the investor's reaction to a higher fee rate, the equilibrium fund allocation to equities increases as the fee increases, thereby decreasing the fund's allocation to the bond and enabling the investor to hold a larger fraction of his bond holdings directly versus indirectly. Thus, as the fee rate increases a larger part of the investor's portfolio is invested directly in the bond, both because the fund tilts its portfolio more towards equity and because the investor allocates less to the fund.⁷

⁷ The argument in the above paragraph implicitly assumes that the fund does not hold more than 100% of its holdings in the bond. While we are not able to categorically rule out this possibility, it is probably highly unlikely.

Multiplying the equilibrium fund portfolio and the investor's equilibrium trading strategy, we get that the investor's effective equity portfolio weights are

$$\left[\frac{\widehat{\pi}_t \phi_t^{\widehat{\pi}}}{W_t^{i\widehat{\pi}}} \right] = [\sigma_t^*]^{-1} \frac{1_{\{\mathcal{B}\}}(\xi_t - \beta_t)}{2}.$$

For a deterministic opportunity set, Corollary 1 shows that the unique solution to the backward stochastic differential equation (23) is such that the vector Y is identically equal to zero; implying that β is identically equal to zero. Thus, the composition of the investor's effective equity portfolio is the same as he would hold if he were able to trade equity directly at no cost, although the investor's overall exposure to equity is smaller by a factor of two.

A stochastic opportunity set, on the other hand, results in a distortion to the equity component of the fund's portfolio. The manager knows that in states of the world where the instantaneous opportunity set is good (i.e., states in which the fund's Sharpe ratio is high) the investor invests a large fraction of his wealth in the fund, thereby generating large fees. This "flow hedging" consideration induces the manager to distort the fund's equity portfolio in such a way as to correlate the investor's wealth process with the Sharpe ratio process.

The fact that the solution to the BSDE (23) is independent from the fee rate implies that investor's effective equity portfolio weights are independent of the fee rate. This in turn implies that the investor's effective overall bond portfolio weight is also independent of the fee rate. Furthermore, multiplying (28) by the fee rate shows that the fraction of the investor's wealth that is paid as management fees is fixed.

Combining the above yields that both the investor's wealth process and the fee process Φ are independent of the fee rate. This in turn implies that equilibrium value functions of both agents are also independent of the fee rate. Thus, in the context of our model, it does not matter whether the investor or the fund manager is the agent setting the management fee. In both cases, the managers ability to select the fund portfolio on the one hand and the investor's ability to adjust the holdings in the fund on the other hand counteract each other so that both agents are equally well off whether one or the other is setting the fee rate.⁸

The first term on the right hand side of (30) is simply the utility that the investor would attain if he was restricted to holding only the bond. The second is the utility gain attributed to being able to trade also the mutual fund. One might have been tempted to conjecture that the fact that the investor is a price taker while the manager is strategic would imply that in equilibrium the investor will be kept to his reservation utility; i.e., the utility he would attain if he could hold only the bond. This argument ignores two important issues. First, in the setting of our model making the investor better off relative to the case where he could invest only in the bond is not necessarily inconsistent with making the manager better off. Second, the investor's ability to move funds in and out of the fund limits the

⁸The model is discontinuous from the left at $\gamma = 0$, as is evident from portfolio holdings in Theorem 1 and the different fee irrelevance results.

managers ability to take advantage of him. This implies that the two agents need to split the surplus, as opposed to the manager being able to expropriate all of the surplus.

If the investor was able to access the financial markets directly without facing any costs, then his expected utility would simply be given by

$$V_i^0 = E \left[\log (W_0^i S_T^0) \right] + E \left[\int_0^T \frac{\|\xi_\tau\|^2}{2} d\tau \right]$$

Thus, the *fraction* of his wealth that the investor would be willing to give up to be able to directly access financial markets is given by

$$p^i = 1 - \exp \left(E \left[\int_0^T \frac{1_{\{\mathcal{B}\}} \|\xi_\tau - \beta_\tau\|^2 - 4 \|\xi_\tau\|^2}{8} d\tau \right] \right) \quad (31)$$

In view of the independence of the equilibrium value functions from the fee rate and given that the market value of the fees to be received is given by $W_0^i [1 - X_0]$ one may be tempted to conjecture that $p^i = 1 - X_0$. To see that this conjecture is false, we focus on the case where the price coefficients are deterministic. Combining Theorem 1 with Corollary 1 yields the following:

Corollary 2 *Assume that the market coefficients are deterministic and bounded functions and let $X : [0, T] \rightarrow [0, 1]$ be as in (26). Then the equilibrium fund portfolio and the investor's equilibrium trading strategy are given by*

$$\hat{\pi}_t = [\sigma_t^*]^{-1} \frac{2\gamma \xi_t}{\|\xi_t\|^2}, \quad \phi_t^{\hat{\pi}} = \|\xi_t\|^2 \left[\frac{W_t^{i\hat{\pi}}}{4\gamma} \right] \quad (32)$$

and the equilibrium expected utilities of each of the two agents are given by

$$V_m := E \left[U \circ I \left(y_{\hat{\pi}} \frac{M_T}{S_T^0} \right) \right] \quad (33)$$

$$V_i := \log (W_0^i S_T^0) + \int_0^T \frac{\|\xi_\tau\|^2}{8} d\tau \quad (34)$$

where $y_{\hat{\pi}}$ solves (14) with $v_0^{\hat{\pi}} = W_0^i (1 - X_0)$. Finally, the manager's equilibrium trading strategy on his own account is given by $\eta^{\hat{\pi}} = [\sigma^*]^{-1} S^0 h$ where the process h is the integrand in the representation of the martingale defined by (15).

Assuming constant, or more generally deterministic, market coefficients and setting $\beta = 0$ in (31) we obtain that a strict inequality

$$p^i = 1 - \exp \left[- \int_0^T \frac{3 \|\xi_\tau\|^2}{8} d\tau \right] > 1 - \exp \left[- \int_0^T \frac{\|\xi_\tau\|^2}{4} d\tau \right] = 1 - X_0$$

holds, and conclude that the presence of delegated portfolio management induces a dead weight cost on the investor.

4.2 Fund Flows

The empirical literature on the relationship between flows and returns (see for example Chevalier and Ellison (1997) or Sirri and Tufano (1998)) has focused on the relationship between past returns and current flows. In some cases, for example Sirri and Tufano (1998), when analyzing this relationship the potential relationship between measured flows and contemporaneous returns has been ignored. In this section we demonstrate that not controlling for contemporaneous returns potentially biases the relationship between lagged returns and flows. Furthermore, we demonstrate that it is important to carefully control for variations in fee rates.

Before defining the different measures of fund flows that we shall consider, we start by using the results of the previous section to express the return on the investor's wealth as a function of the equilibrium fund return.

Lemma 3 *(F, W^i) denote respectively the equilibrium fund value process and the investor's equilibrium wealth process and assume that the coefficients of the financial market are constant. Then we have*

$$\left[\frac{W_{t+\Delta}^i}{W_t^i} \right] = e^{\alpha\Delta} \left[\frac{F_{t+\Delta}}{F_t} \right]^{\frac{\|\xi\|^2}{4\gamma}} \quad (35)$$

for every non negative time step where we have set $\alpha := (r + \frac{\gamma}{2}) \left(1 - \frac{\|\xi\|^2}{4\gamma}\right)$.

Let Δ be an arbitrary but fixed time step. Denote by $R_{t,\Delta}$ the return that the investor would obtain from investing in the fund between dates t and $t + \Delta$ if management fees were waived, that is

$$R_{t,\Delta} := e^{\gamma\Delta} \left[\frac{F_{t+\Delta}}{F_t} \right]$$

We consider two alternative measures for mutual fund flows. The first is defined by

$$\rho_1 := \left[\frac{\phi_{t+\Delta} - R_{t,\Delta}\phi_t}{R_{t,\Delta}\phi_t} \right] = e^{(\alpha - \frac{1}{4}\|\xi\|^2)\Delta} R_{t,\Delta}^{\frac{\|\xi\|^2}{4\gamma} - 1} - 1 \quad (36)$$

while the second, which is the one typically used in empirical studies (see for example Chevalier and Ellison (1997)), is defined by

$$\rho_2 := \left[\frac{\phi_{t+\Delta} - R_{t,\Delta}\phi_t}{\phi_t} \right] = \rho_1 R_{t,\Delta}. \quad (37)$$

It is evident from (36) and (37) that both measures are sensitive to variations in the fee rate. By analyzing ρ_1 and its derivatives as a function of both the total return on the fund and the fee rate, the following proposition is obtained

Proposition 4

1. *As a function of the total return on the fund, the measure ρ_1 is decreasing if $\|\xi\|^2 < 4\gamma$ and increasing otherwise.*

2. *As a function of the total return on the fund, the measure ρ_1 is convex if either $\|\xi\|^2 < 4\gamma$ or $8\gamma \leq \|\xi\|^2$ and concave otherwise.*
3. *For a given level of total return on the fund, the measure ρ_1 is increasing in the fee rate if $2\gamma^2 \geq (\Delta^{-1} \log(R_{t,\Delta}) - r) \|\xi\|^2$ and decreasing otherwise.*

Given constant market coefficients, both the investor and the fund hold fixed portfolios. If the investor is non levered then he holds the bond both directly and indirectly through his holdings of the fund. Therefore, the bond constitutes a larger fraction of his portfolio relative to that of the fund. When the investor is levered his portfolio is tilted more towards equities relative to the fund portfolio. As can be seen from Corollary 2, the investor holds a levered position in the fund if and only if $4\gamma < \|\xi\|^2$.

The direction of the wedge between the investor's portfolio and the fund portfolio determines whether the contemporaneous relationship between flows and returns will be increasing or decreasing. For a non levered investor: when the realized return on the fund is higher than the return on the bond, as fund returns increase the investor withdraws money from the fund in order to maintain his fixed portfolio weights. On the other hand, a positive inflow is required to maintain fixed weights following a fund return that is lower than the return on the bond. The opposite occurs when the investor is levered. Furthermore, for both levered and unlevered investor's, the more extreme the fund returns are the more pronounced the rebalancing will be.

The third part of the proposition is a consequence of the fact that a higher fee rate implies that a larger part of the investor's portfolio is held directly in the bond. For example, following a negative fund return the investor needs to inject money into the fund in order to rebalance his portfolio back to the optimal composition. Holding this return level fixed: a higher fee rate implies a higher return on the investor's portfolio (i.e. less negative) which in turn implies less rebalancing on the part of the investor. While the condition in the third part of the proposition depends both on the fund return and the fee rate, it is evident that there are three regions: for sufficiently low fund returns the flow is an increasing function of the fee rate, for sufficiently high returns the flow is a decreasing function of the fee rate, and for intermediate values the relationship is inverse U-shaped.

By construction, we have $\rho_2 = \rho_1 R$ so that ρ_2 is positive if and only if ρ_1 is. For positive flows, ρ_2 will give a higher (lower) value than ρ_1 if the total return on the fund is above (below) 1. For negative flows ρ_2 will give a lower (higher) value than ρ_1 if the total return on the fund is above (below) 1.⁹ Furthermore, by analyzing ρ_2 and its derivatives the following proposition is obtained

Proposition 5

⁹When analyzing the relationship between lagged returns and flows it is typically the case that flows are monotonic in lagged returns. Thus, the relationship between lagged returns and flows will typically be less convex for ρ_1 than for ρ_2 . The difference may be significant if flows and past returns are measured over a long horizon. Both in Chevalier and Ellison (1997) and Sirri and Tufano (1998) flows and past returns are measured over a period of a year.

1. As a function of the total return on the fund, the measure ρ_2 is inverse U-shaped if $\|\xi\|^2 < 4\gamma$ and U-shaped otherwise.
2. As a function of the total return on the fund, the measure ρ_2 is convex if and only if $4\gamma \leq \|\xi\|^2$.
3. For a given level of total return on the fund, the measure ρ_2 is increasing in fee rate if $2\gamma^2 \geq (\Delta^{-1} \log(R_{t,\Delta}) - r) \|\xi\|^2$ and decreasing otherwise.

The third part of the two propositions is the same, but the first and second are not. A disadvantage of using ρ_2 , as can be deduced by comparing the first part of the two propositions, is that it distorts the flow performance relationship. In particular, with ρ_2 it is no longer the case that the flow performance relationship is monotone for a given fee rate. Instead, it is either U-shaped or inverse U-shaped. As a result of this non monotonicity, in order to control for the impact of contemporaneous returns on measured flows it is no longer sufficient to know the relevant region of the fee rate, as is the case with ρ_1 , it is also necessary to know the fund total rate of return at which the derivative of the flow measure ρ_2 switches sign.¹⁰

4.3 Restricting the Fund to Holding Only Equity

Given that the fund has an advantage in accessing equity relative to the investor but does not have an advantage in accessing the bond, it is natural to also consider the case in which the fund is restricted to holding only equity. For simplicity of exposition, we restrict the analysis of this section to the case with constant market coefficients.

Given constant market coefficients, for a given fee rate the fund's optimal portfolio is still the one that maximizes its instantaneous net of fees Sharpe ratio, with the additional restriction that it can trade only equity. Solving the constrained optimization program

$$\sup_{\pi \in \mathbb{R}^n} \left\{ \frac{[\pi^* \sigma \xi - \gamma]^+}{\|\sigma^* \pi\|^2} \text{ s.t. } \pi^* \mathbf{1} = 1 \right\} \quad (38)$$

and plugging the result into the formulae of Proposition 1 and 2, we obtain the following counterpart to Corollary 2.¹¹

Proposition 6 *Assume that the market coefficients are constant and that the fund is restricted from investing in the bond. Then the equilibrium fund portfolio and the investor's equilibrium trading strategy are given by*

$$\hat{\pi}_t = \pi(\gamma) := [\sigma^*]^{-1} \frac{\xi - \lambda(\gamma) A}{\xi^* A - \lambda(\gamma) \|A\|^2} \quad (39)$$

$$\phi_t^{\hat{\pi}} = \left(\xi^* A - \lambda(\gamma) \|A\|^2 \right) \left[\frac{W_t^{i\hat{\pi}}}{2} \right] \quad (40)$$

¹⁰ Another disadvantage of ρ_2 has been pointed out by Berk and Green (2002).

¹¹ Note that this optimization problem solves for the tangency portfolio in an economy with an expected return vector $a - \gamma$ and covariance matrix $[\sigma^* \sigma]$.

where $\lambda^{(\gamma)} := \gamma - \|\xi - \gamma A\| / \|A\|$ and $A := [\sigma^{-1}\mathbf{1}]$. The equilibrium expected utilities of each of the two agents are respectively given by

$$V_m = E \left[U \circ I \left(y_{\hat{\pi}} \frac{M_T}{S_T^0} \right) \right] \quad (41)$$

$$V_i = \log (W_0^i S_T^0) + \frac{1}{8} \left\| \xi - \lambda^{(\gamma)} A \right\|^2 T \quad (42)$$

where $y_{\hat{\pi}}$ solves (14) with

$$v_0^{\hat{\pi}} = W_0^i \left[1 - \exp \left(-\frac{\gamma}{2} [\xi - \lambda^{(\gamma)} A]^* A T \right) \right]. \quad (43)$$

Finally, the manager's equilibrium trading strategy on his own account is given by $\eta^{\hat{\pi}} = [\sigma^*]^{-1} S^0 h$ where the process h is the integrand in the representation of the martingale defined by (15).

Using the fact that the fund is restricted to trading only equity implies that the instantaneous net of fees Sharpe ratio can be written as

$$\vartheta_t^\pi = \frac{\pi_t^* a}{\|\sigma^* \pi_t\|^2} - \frac{r + \gamma}{\|\sigma^* \pi_t\|^2}.$$

Thus, higher fee rates will induce the manager to shift the fund's portfolio towards more volatile portfolios; similar to the case where the fund is allowed to trade also in the bond. The difference between the two is the mechanism used to increase the volatility. When the manager is allowed to also trade in the bond, the higher volatility is a result of increasing leverage without changing the composition of the equity portion of his portfolio. When the manager is restricted to trading only equity, he can not change the fund's leverage so he increases volatility by changing the composition of the equity portfolio. As a result, it is no longer the case that the investor's effective equity portfolio is independent of the fee rate; as can be seen by multiplying the equilibrium fund portfolio and the investor's equilibrium trading strategy.

With fixed market coefficients, both when the fund is allowed to trade in the bond and when it is prohibited to, the fund's optimal portfolio is the one that maximizes the relevant net of fees Sharpe ratio. This Sharpe ratio will be at least as high in the case where it is allowed to trade also in the bond, since in that case holding only equity is a feasible fund strategy. As a result, when the fund is restricted to trading only equity the manager's welfare would be at most the one obtained when the fund is unrestricted.

A direct optimization of the manager's equilibrium expected utility over the instantaneous fee rate then yields the following¹²

¹²This corollary could also be obtained directly from Corollary 2 by solving for the fee rate that implies that the fund is fully invested in equity.

Corollary 3 *Assume that the market coefficients are constant and that the fund is restricted from investing in the bond. Then the instantaneous fee rate that maximizes the manager’s welfare is given by*

$$\hat{\gamma} := \frac{\|\xi\|^2}{2(\xi^*A)} = \frac{\|\xi\|^2}{2\xi^*[\sigma^{-1}\mathbf{1}]}.$$
 (44)

The corresponding equilibrium fund portfolio and investor’s equilibrium trading strategy are given by

$$\hat{\pi} = \pi(\hat{\gamma}) = [\sigma^*]^{-1} \frac{\xi}{(\xi^*A)}, \quad \phi_t^{\hat{\pi}} = (\xi^*A) \left[\frac{W_t^{i\hat{\pi}}}{2} \right].$$
 (45)

Finally, the equilibrium expected utilities of the two agents and the manager’s equilibrium trading strategy on his own account are as in Corollary 2.

The welfares of both agents under this Pareto optimal fee rate are identical to that in Corollary 2. Therefore, an intuitive interpretation of the fee rate irrelevance result of Section 4.1 is as follows. The fund has an advantage in that it is able to access the risky assets where as the investor can not. On the other hand it does not have any advantage in accessing the bond. As such, the “optimal equity access tax” that it can extract from the investor can be found by restricting the fund to trading only equity, but allowing it to set the fee rate. Allowing the fund to trade also in the bond enables it to extract the “optimal equity access tax” irrespective of the fee rate. It does so by varying its bond stock mix so that the investor’s effective exposure to equity as well as the fees paid remain the same as under the strategy that implements the “optimal equity access tax”. For higher (lower) fee rates it tilts its portfolio toward equity (the bond), taking into account the fact that a higher (lower) fee rate implies that the investor, holding all else equal, will be willing to invest more (less) in the fund. Thus, for every fee rate there is a corresponding fund portfolio that implements the optimal tax. Theorem 1 shows that this intuition also holds for a stochastic opportunity set.

Given that the welfares of both agents under the fee rate of Corollary 3 are the same as in Corollary 2, when the fund manager is the one setting the fee rate both agents are indifferent whether the fund is restricted to trading only equity or not. Differentiating (42) with respect to the fee rate shows that the investor’s welfare is a decreasing function of the fee rate. Furthermore, (43) shows that the manager’s welfare is an inverse U-shaped function of the fee rate, with a maximum at the fee rate of Corollary 3. Thus, if the investor gets to set the fee rate the two agents have opposing views on whether to allow the fund to trade only equity or enable it to also trade in the bond. Specifically, when the investor gets to set the fee rate he will drive down the manager to his reservation utility.

5 Relaxing the Inaccessibility Assumption

We now extend the model of the previous sections in order to allow the investor to have direct access to some of the stocks traded on the market. To simplify the exposition, let us

start by introducing some notation. Let m be the number of stocks that the investor can trade and without loss of generality assume that these stocks are the first m components of the stock price process. Let $\bar{\sigma}$ denote the volatility matrix of these stocks. For an arbitrary fund portfolio process, define the matrix valued processes

$$\Lambda_t^\pi := \begin{bmatrix} \pi_t^* \sigma_t \\ \bar{\sigma}_t \end{bmatrix}, \quad \Sigma_t^\pi := \Lambda_t^\pi [\Lambda_t^\pi]^*$$

and let Π_f^* denote the set of fund portfolio processes which are such that the variance covariance matrix Σ^π is invertible. Note that if this matrix is not invertible, then the fund is a linear combination of the m stocks available to the investor and is thus strictly dominated.

5.1 Investor's Optimal Strategy

Let W_0^i denote the investor's initial capital and assume that he trades in the available risky assets according to a \mathbb{R}^{m+1} -valued process ψ with non negative first component $\phi = \psi^1$. Given a fund portfolio process, the investor's wealth process satisfies the dynamic budget constraint

$$\begin{aligned} dW_t^i &= \left[r_t W_t^i - \gamma \phi_t \right] dt + \psi_t^* \Lambda_t^\pi \left[dB_t + \xi_t dt \right] \\ &= r_t W_t^i dt + \psi_t^* \Lambda_t^\pi \left[dB_t + \xi_t^\pi dt \right] \end{aligned} \quad (46)$$

where ξ^π denotes the modified risk premium process associated with the risky assets to which the investor has access, that is

$$\xi_t^\pi := \xi_t - \gamma [\Lambda_t^\pi]^* [\Sigma_t^\pi]^{-1} \mathbf{e}.$$

where $\mathbf{e} = (1, 0, \dots, 0) \in \mathbb{R}^{m+1}$. As in the previous sections, the investor's best response problem is to maximize the expected logarithmic utility of his terminal wealth over the set Θ_i^π of trading strategies which satisfy the no short selling constraint on the fund as well as (6) and are such that the corresponding solution to (46) is a non negative process.

In order to facilitate the description of the investor's best response trading strategy, let us first fix some more notation. For any fund portfolio process $\pi \in \Pi_f^*$ we define a pair of vector valued processes by setting

$$C_t^\pi := [\Sigma_t^\pi]^{-1} \Lambda_t^\pi \xi_t^\pi, \quad A_t^\pi := \frac{[\Sigma_t^\pi]^{-1} \mathbf{e}}{\mathbf{e}^* [\Sigma_t^\pi]^{-1} \mathbf{e}} \quad (47)$$

where \mathbf{e} is as above. After these rather lengthy preparations, we are now ready to state the main result of this section.

Proposition 7 *Let $\pi \in \Pi_f^*$ denote an arbitrary fund portfolio process. Then the investor's best response trading strategy is given by*

$$\psi_t^{i\pi} = W_t^{i\pi} \left(C_t^\pi + A_t^\pi [\mathbf{e}^* C_t^\pi]^- \right)$$

where $W^{i\pi}$ is the corresponding solution to (46). In particular, the investor's best response investment in the fund is given by $\phi^\pi := [\psi^{i\pi}]^1 = W^{i\pi}[\mathbf{e}^*C^\pi]^+$.

The intuition for the above result is as follows. First, the investor determines the trading strategy that he would adopt if he was not subject to the no short selling constraint on the fund. This yields the term $W^{i\pi}C^\pi$. When the no short selling constraint on the mutual fund binds (that is when the quantity \mathbf{e}^*C^π is negative), the investor sets his position in the fund to zero and projects the negative fraction of his wealth that he would have liked to invest in the fund on the m available stocks according to the vector A^π . This yields the second term in the expression for the best response trading strategy.

5.2 Optimal Fund Portfolio

As in Section 3.2.2 the optimal fund portfolio process is that which maximizes the initial market value of the management fees, and is obtained by solving the stochastic control problem with value function

$$\widehat{v}_0 := \sup_{\pi \in \Pi_f^*} v_0^\pi = \sup_{\pi \in \Pi_f^*} E_Q \left[\int_0^T \frac{\gamma \phi_\tau^\pi}{S_\tau^0} d\tau \right]. \quad (48)$$

Corresponding to Lemma 1 we now have

Lemma 4 *Fix an arbitrary fund portfolio process $\pi \in \Pi_f^*$ and let $W^{i\pi}$ denote the corresponding best response wealth process. Then the process defined by*

$$M_t^\pi := \exp \left(\int_0^t \gamma [\mathbf{e}^*C_\tau^\pi]^+ d\tau \right) \left[\frac{W_t^{i\pi}}{S_t^0 W_0^i} \right]$$

with the process C^π as in (47), is a strictly positive and uniformly integrable martingale under the risk neutral probability measure.

Combining the result of Lemma 4 with Girsanov's theorem we have that for an arbitrary fund portfolio process $\pi \in \Pi_f^*$ the formula $Q^\pi(A) := E_Q[1_A M_T^\pi]$ defines a probability measure which is equivalent to the risk neutral measure and under which the process

$$Z_t^\pi := Z_t + \int_0^t [\Lambda_\tau^\pi]^* \left(C_\tau^\pi + A_\tau^\pi [\mathbf{e}^*C_\tau^\pi]^- \right) d\tau$$

is an n -dimensional standard Brownian motion. An argument similar to that which lead to (20) then shows that (48) can be written as

$$\widehat{v}_0 = W_0^i \left\{ 1 - \inf_{\pi \in \Pi_f^*} E_{Q^\pi} \left[e^{-\gamma \int_0^T [\mathbf{e}^*C_\tau^\pi]^+ d\tau} \right] \right\}.$$

As in Section 3.2.2, one can convert this problem into one which optimizes over a family of backward stochastic equations. Unfortunately, the associated drift optimization cannot

be solved explicitly, unless the coefficients of the model are deterministic. To gain some insights on the impact of the ability of the investor to trade directly in a subset of the risky assets on the manager's portfolio choice, we henceforth restrict the analysis to the case of deterministic coefficients.

Let $\hat{\pi}$ denote an arbitrary fund portfolio with values in $\arg \max_{\pi} [\mathbf{e}^* C_t^{\pi}]^+$. As is easily seen from measurable selection theorems, any such fund portfolio is a deterministic process and it follows that

$$\hat{v}_0 = v_0^{\hat{\pi}} = W_0^i \left[1 - e^{-\int_0^T \gamma [\mathbf{e}^* C_{\tau}^{\hat{\pi}}]^+ d\tau} \right]$$

This shows that any function with values in $\arg \max_{\pi} [\mathbf{e}^* C_t^{\pi}]^+$ is an optimal fund portfolio. In order to facilitate the description of this set, recall that we denote by $\bar{\sigma}$ the volatility matrix of the available stocks and set

$$\Omega_t = \mathbb{I}_n - \Omega_t^0 := \mathbb{I}_n - \bar{\sigma}_t^* [\bar{\sigma}_t \bar{\sigma}_t^*]^{-1} \bar{\sigma}_t$$

where \mathbb{I}_n denotes the n -dimensional unit matrix. Let further $\Omega_t^{\perp} := \text{Ker}(\Omega_t)$ denote the null space of the matrix Ω_t and introduce the set

$$\mathcal{S}_t := [\sigma_t^*]^{-1} \left(\Omega_t^{\perp} + \frac{2\gamma \xi_t}{\xi_t^* \Omega_t \xi_t} \right).$$

Since the rows of the matrix $\bar{\sigma}$ are linearly independent, the square matrix Ω_t has rank $n - m$ and it therefore follows from standard linear algebra (see for example Smith (1984)) that for each t the set \mathcal{S}_t is a vector subspace of dimension m . The following proposition provides an explicit characterization of the optimal fund portfolio in the case of constant market coefficients.

Proposition 8 *Assume that the market coefficients are deterministic. Then the set of optimal fund portfolios is the set of non zero \mathcal{S} -valued functions.*

The above result shows that because the investor has access to m risky assets on top of the fund, the manager has m degrees of freedom in choosing the fund portfolio. In particular, the initial market value of the management fees does not depend on the choice of the \mathcal{S} -valued function $\hat{\pi}$ and is given by $\hat{v}_0 = W_0^i [1 - X_0]$ where

$$X_t := \exp \left[- \int_t^T \frac{\xi_{\tau}^* \Omega_{\tau} \xi_{\tau}}{4} d\tau \right]. \quad (49)$$

Given that the manager has m degrees of freedom in determining the fund portfolio, it is natural to ask whether there exists an optimal portfolio process which entails the fund to trade only in the bond and the $n - m$ risky assets that the investor cannot trade in. The following result provides an explicit characterization of this particular optimal portfolio process.

Corollary 4 *Assume that the market coefficients are deterministic and bounded functions. Then the process defined by*

$$\widehat{\pi}_t^k := 1_{\{k>m\}} \frac{2\gamma[(\sigma_t^*)^{-1}\xi_t]^k}{\xi_t^*\Omega_t\xi}, \quad 1 \leq k \leq n$$

is the unique optimal fund portfolio with the property that the fund does not invest in any of the stocks that the investor can access directly.

5.3 Equilibrium

Gathering the previous results, we now obtain a complete description of the equilibrium under the assumption of deterministic market coefficients.

Theorem 2 *Assume that the market coefficients are deterministic and bounded functions and let $X : [0, T] \rightarrow [0, 1)$ be as in (49). Then the equilibrium fund portfolio and the investor's equilibrium trading strategy in the fund are given by*

$$\widehat{\pi}_t = [\sigma_t^*]^{-1} \left[L_t + \frac{2\gamma\xi_t}{\xi_t^*\Omega_t\xi_t} \right], \quad (50)$$

$$\phi_t^{\widehat{\pi}} = \xi_t^*\Omega_t\xi_t \left[\frac{W_t^{i\widehat{\pi}}}{4\gamma} \right] \quad (51)$$

where L_t is any vector in the null space of Ω_t . The equilibrium expected utilities of each of the two agents are given by

$$V_m := E \left[U \circ I \left(y_{\widehat{\pi}} \frac{M_T}{S_T^0} \right) \right] \quad (52)$$

$$V_i := \log(W_0^i S_T^0) + \int_0^T \left[\frac{\|\xi_\tau\|^2}{8} + \frac{3\xi_\tau^*\Omega_t\xi_\tau}{8} \right] d\tau \quad (53)$$

where $y_{\widehat{\pi}}$ solves (14) with $v_0^{\widehat{\pi}} = W_0^i(1 - X_0)$. Finally, the manager's equilibrium trading strategy on his own account is given by $\eta^{\widehat{\pi}} = [\sigma^*]^{-1}S^0h$ where the process h is the integrand in the representation of the martingale defined by (15).

The results are consistent with the ones obtained under the assumption that the investor is restricted from trading equity. First, the investor's holding of the fund is inversely related to the fee rate. Second, and as can be seen from multiplying $\phi^{\widehat{\pi}}$ and $\widehat{\pi}$, the investor's effective equity portfolio as well as the fraction of his wealth that is invested in the bond is independent of the fee rate. Third, increasing the fee rate results in the investor holding a larger part of his bond position directly versus indirectly through his holding of the fund. Forth, the cumulative fee process $\Phi^{\widehat{\pi}}$ is independent of the fee rate, as can be seen from multiplying the fee rate by the investor's holdings in the fund. Fifth, combining the second and the forth above implies that the investor's wealth process is independent of the fee rate, so that his value function should be independent of the fee rate; as it is. Sixth, combining

the fourth and the fifth implies that the manager's value function should be independent of the fee rate; as it is. Thus, both value functions are again independent from the fee rate, even though the investor can trade some of the risky assets directly.

As can be expected, allowing the investor to trade in some of the risky assets allows him to increase his expected utility relative to the case where he can get equity exposure only through trading the fund. Comparing Equations (34) and (53) shows that direct access to m stocks increases the investor's welfare by the quantity

$$u(m) := \int_0^T \left[\frac{3\xi_\tau^* \Omega_\tau^0 \xi_\tau}{8} \right] d\tau \quad (54)$$

which is non negative since the matrix Ω_t^0 is positive semidefinite. Intuitively, the increase in the investor's welfare compared to the base case of Section 4.1 should be monotone increasing in the number of stocks that the investor can trade in. The following lemma confirms this intuition

Lemma 5 *Assume that the market coefficients are deterministic and bounded functions. Then the function $u : \{0, 1, \dots, m\} \rightarrow \mathbb{R}_+$ defined by (54) is increasing.*

Specifically, when the investor cannot trade any of the stocks directly, that is when $m = 0$, the matrix Ω_t^0 vanishes and we recover the results of Section 4.1. On the other hand, when the investor has direct access to all of the risky assets, the matrix Ω_t^0 becomes the unit matrix and hence the value function of the investor becomes

$$V_i = \log(W_0^i S_T^0) + \int_0^T \left[\frac{\|\xi_\tau\|^2}{2} \right] d\tau,$$

which is the expected utility obtained by a *log* investor who has complete access to the financial market. In this case, the fund becomes a redundant asset which the investor does not want to trade because of the non zero fee rate, and as a result the manager is not able to extract any fees from the investor.

6 Conclusion

In this paper we have analyzed the impact of the ability of investors to dynamically determine their holdings in a fund on a fund's dynamic portfolio choice decisions. To the best of our knowledge, this is the first paper to theoretically analyze the impact of fund flows on a fund's portfolio choice decisions in a dynamic setting in which both the fund's portfolio decisions and the flows are determined endogenously.

Our model predicts a positive relationship between a fund's proportional fee rate and a fund's equity exposure, or alternatively a fund's volatility. We show that while funds that have higher proportional fee rates are expected to tilt their portfolio towards stocks relative to funds with lower fee rates, it is not necessarily true that the dollar amount held in equities by higher fee rate funds will also be higher. In equilibrium, higher fees also imply

that investors will allocate less of their wealth to these funds. In the context of our model, the two effects exactly offset each other, so that in equilibrium the amount of equity held by a fund is independent of its fee rate. Furthermore, we show that the combination of the fund’s relative advantage in accessing equity with the investor’s ability to dynamically move money in and out of the fund implies the existence of a fee rate independent “equity access tax” that the fund can extract from the investor. The independence between the fee rate and the agents welfares in turn implies that it does not matter who gets to set the fee rate, the investors or the fund.

In the current paper, the choice of a logarithmic utility function for the investor was made for tractability. Given that we wanted to focus on how a fund’s portfolio choice decision is impacted by the presence of dynamic flows, a natural simplifying assumption was to make the investor myopic, so that only the fund manager will have any kind of hedging demands. As a result, when the investment opportunity set is stationary we obtain that there are no hedging demands by either the investor or the fund, and the fund portfolio is chosen so that to maximize the investor’s net of fees Sharpe ratio. On the other hand, a non stationary investment opportunity set induces “flow hedging” demands on the part of the fund. Specifically, the fund manager takes advantage of the fact that she knows that a *log* investor determines the fraction of his wealth to be invested in the fund according to the fund’s net of fees Sharpe ratio. Thus, she distorts the fund’s portfolio so that to positively correlate between the investor’s future wealth and future fund Sharpe ratios. This strategy guarantees that the investor will be wealthy in those states where he will be willing to delegate a large fraction of his wealth to the fund, thereby generating large fees.

Allowing for utility functions other than $\log(\cdot)$ for the investor would introduce hedging considerations also on his part. In such a setting there will be both “usual” hedging considerations on the part of the investor and “flow hedging” considerations on the part of the fund. This would probably make the problem analytically intractable; requiring a non trivial numerical solution.

We have also demonstrated the importance of controlling for the contemporaneous flow performance relationship in studies analyzing the relationship between lagged returns and flows. Specifically, we have derived explicit predictions related to the contemporaneous performance flow relationship, and have highlighted the importance of carefully controlling for the potentially non linear impact that different fee rates may have on the observed performance flow relationship. We have also pointed out to a disadvantage of using the commonly used flow measure relative to another closely defined measure.

While we view this paper as an important first step in understanding the dynamic portfolio choice decisions made by fund managers, there are some important extensions that are left for future work. First, while a proportional fee is the most commonly used compensation contract in the mutual fund industry, there are some funds that also use fulcrum performance fees as part of their compensation contract. Furthermore, both pension funds and hedge funds typically have an asymmetric performance fee component; whereby the fund manager shares in the upside but not in the down side. The impact of such

performance fees on a fund's portfolio choice has previously been analyzed either in a setting where at the initial date the manager receives an exogenously specified amount of money to manage for an exogenously specified investment period (see for example Grinblatt and Titman (1989) or Carpenter (2000)), or at most in a setting in which the investor is restricted to making his or her fund allocation decision only at the initial date (see Cuoco and Kaniel (2001)). One could attempt to extend that analysis to a dynamic setting similar to the one in the current paper. Second, in some cases, especially in hedge funds, the manager commits part of her capital to investing in the fund she manages. Thus, portfolio decisions that are made by the fund might adversely effect the manager's own portfolio composition. In such a setting, it will no longer be true that the fund's portfolio is determined so that to maximize the present value of the fees to be received. Furthermore, disallowing the manager to hedge the management fees creates a stronger link between the dynamic optimization problems of the investor and the manager. This stronger link makes the problem considerably harder to solve. Finally, explicitly incorporating asymmetric information between the manager and the investor and or allowing for multiple competing mutual funds are both important challenging extensions.

A Proofs

Proof of Proposition 1. Let $\pi \in \Pi_f$ be an arbitrary admissible fund portfolio process, ϕ^π be the candidate optimal strategy defined by (11) and $W_{i\pi}$ denote the corresponding wealth process. The fact that ϕ^π is an admissible trading strategy for the investor follows from the result of Lemma A.1 which also shows that the corresponding wealth process is strictly positive.

In order to verify that ϕ^π identifies the investor's best response trading strategy, let ϕ be an arbitrary admissible trading strategy for the investor given the fund portfolio process and denote by W^i the corresponding wealth process. Using Itô's product rule in conjunction with (3) and (11) we have

$$\begin{aligned} d \left[\frac{W_t^i}{W_t^{i\pi}} \right] &= \frac{dW_t^i}{W_t^{i\pi}} + W_t^i d \left[\frac{1}{W_t^{i\pi}} \right] + d \left\langle W^i, \frac{1}{W^{i\pi}} \right\rangle_t \\ &= \left(\phi_t - [\vartheta_t^\pi]^+ W_t^i \right) \frac{\pi_t^* \sigma_t}{W_t^{i\pi}} dB_t - \frac{\|\sigma_t^* \pi_t\|^2}{W_t^{i\pi}} \left([\vartheta_t^\pi]^- \phi_t \right) dt \end{aligned}$$

where ϑ^π is the process defined by (10). Observing that both ϕ and W_i/W_i^π are non negative processes and applying Fatou's lemma we conclude that the latter process is a supermartingale under the objective measure. Using this in conjunction with the concavity and increase of the logarithm we obtain that

$$\begin{aligned} E \left[\log(W_T^i) \right] - E \left[\log(W_T^{i\pi}) \right] &= E \left[\log \left(\frac{W_T^i}{W_T^{i\pi}} \right) \right] \\ &\leq \log \left(E \left[\frac{W_T^i}{W_T^{i\pi}} \right] \right) \leq \log \left(\frac{W_0^i}{W_0^{i\pi}} \right) = 0 \end{aligned}$$

holds and it now only remains to observe that the investor's admissible trading strategy $\phi \in \Theta_i^\pi$ was arbitrary in order to conclude our proof. \square

Lemma A.1 *For every fund portfolio process $\pi \in \Pi_f$, the cumulative fee process Φ^π defined by (12) is integrable under the risk neutral probability measure.*

Proof. The process $W^{i\pi}$ being non negative we deduce from (12) that Φ^π is increasing and since its initial value is equal to zero, it is also non negative. Let f be the progressively measurable mapping defined by

$$f(t, \pi_t) := [\vartheta_t^\pi]^+ = \frac{[\pi_t^* \sigma_t \xi_t - \gamma]^+}{\|\sigma_t^* \pi_t\|^2}. \quad (\text{A.1})$$

By definition, we have $f(t, \pi_t) \leq \sup_{\pi \in \mathbb{R}^n} f(t, \pi)$ for every $\pi \in \Pi_f$. A simple computation shows that the supremum is uniquely attained by the progressively measurable process

$$\tilde{\pi}_t := \arg \max_{\pi \in \mathbb{R}^n} f(t, \pi) = [\sigma_t^*]^{-1} \frac{2\gamma \xi_t}{\|\xi_t\|^2}$$

and that the corresponding value is given by $(4\gamma)^{-1} \|\xi_t\|^2$. Let now $\ell \in \mathbb{R}_+$ denote a uniform bound on the norm of the risk premium process. The increase of Φ^π and the boundedness of the discount factor imply that we have

$$\begin{aligned} E_Q \left[\sup_{0 \leq t \leq T} \Phi_t^\pi \right] &= E_Q [\Phi_T^\pi] \leq c_1 \cdot E_Q \left[\int_0^T \frac{d\Phi_\tau^\pi}{S_\tau^0} \right] \\ &\leq c_2 \cdot E_Q \left[\int_0^T \frac{W_\tau^{i\pi} \|\xi_\tau\|^2}{S_\tau^0} d\tau \right] \\ &\leq c_2 \cdot E_Q \left[\int_0^T \frac{W_\tau^{i\pi} \ell}{S_\tau^0} d\tau \right] \leq c_3 \cdot W_0^i \end{aligned}$$

for some non negative constants $(c_i)_{i=1}^3$ where the third inequality follows from Fubini's theorem and the fact that the investor's discounted wealth process is a non negative supermartingale under the risk neutral probability measure. \square

Proof of Proposition 2. Let y_π be as in the statement. The non negative random variable $W_T^{m\pi}$ being feasible by construction all we have to prove is that it is optimal for the manager's problem. By concavity of the manager's utility function we have that

$$\begin{aligned} E[U(W_T)] - E[U(W_T^{m\pi})] &\leq E[U'(W_T^{m\pi})(W_T - W_T^{m\pi})] \\ &= y_\pi \left(E_Q \left[\frac{W_T}{S_T^0} \right] - v_0^\pi - W_0^m \right) \leq 0 \end{aligned}$$

holds for every feasible random variable where the second inequality follows from the budget constraint (13) and the fact that y_π is strictly positive. The above expression shows that the random variable of (14) is optimal for the manager's problem and it only remains to observe that uniqueness follows from the strict concavity of the manager's utility function to complete the proof. \square

Proof of Lemma 1. Applying Itô's lemma to the investor's best response wealth process and using (18), we obtain that the non negative process M^π is the stochastic exponential of the local martingale

$$N_t^\pi := \int_0^t f(\tau, \pi_\tau) \pi_\tau^* \sigma_\tau dZ_\tau,$$

hence a local martingale itself. Coming back to the definition of the progressively measurable mapping f in (A.1), it is easily seen that we have

$$\langle N^\pi \rangle_t = \int_0^t \|f(\tau, \pi_\tau) \sigma_\tau^* \pi_\tau\|^2 d\tau \leq \int_0^t \|\xi_\tau\|^2 d\tau \leq L$$

for some $L \in \mathbb{R}_+$ where the last inequality follows from the boundedness of the risk premium process. The above expression readily implies that the local martingale M^π satisfies the Novikov condition and our proof is complete. \square

Proof of Lemma 2. Let $\ell \in \mathbb{R}_+$ denote a uniform bound on the Euclidean norm of the risk premium process. Coming back to map \hat{g} is almost surely continuous in (t, x, y) . On the other hand, an easy computation using the Cauchy-Schwartz inequality shows that

$$|\hat{g}(t, x, y)| \leq \|\xi_t \sqrt{x}\|^2 \leq \ell|x|$$

holds almost surely for all (t, x, y) and it now only remains to apply the results of Lepeltier & San Martin (see Theorems B.3 and B.4 in Appendix B) to obtain the existence of maximal solution with the desired properties. \square

Proof of Proposition 3. The solution to (23) being continuous, the process $\hat{\pi}$ is locally bounded. On the other hand, applying Itô's lemma we have that the solution $F = F^{\hat{\pi}}$ to the stochastic differential equation (2) is given by

$$F_t = S_t^0 \exp \left[-\frac{1}{2} \int_0^t \|\sigma_\tau^* \hat{\pi}_\tau\|^2 ds + \int_0^t \hat{\pi}_\tau^* \sigma_\tau dZ_\tau \right].$$

The process F/S^0 is thus a non negative local martingale under the risk neutral probability measure and it now follows from the non negativity of the discount factor that $\hat{\pi}$ is an admissible fund portfolio process. Using (23) in conjunction with the uniqueness of the solution to (21) and the maximality of X in the set of solutions to (23) we obtain that

$$\operatorname{ess\,inf}_{\pi \in \Pi_f} X_t^\pi \leq X_t^{\hat{\pi}} \leq X_t \tag{A.2}$$

holds almost everywhere. On the other hand, the definition of the map \hat{g} and Theorem B.4 show that we have $X \leq X^\pi$ for every admissible fund portfolio process and taking essential infimum over $\pi \in \Pi_f$ on both sides we conclude that equality holds in (A.2). Using this fact in conjunction with (17) and (19) it is now easily seen that

$$v_0^{\hat{\pi}} = W_0^i(1 - X_0^{\hat{\pi}}) = W_0^i \left(1 - \operatorname{ess\,inf}_{\pi \in \Pi_f} X_0^\pi \right) \geq W_0^i(1 - X_0^\pi) = v_0^\pi$$

holds for every admissible fund portfolio process. This implies that $\hat{\pi}$ constitutes the optimal fund portfolio process and our proof is complete. \square

Proof of Corollary 1. Assume that the risk premium satisfies the conditions of the statement and consider the candidate solution $(X, 0)$. The deterministic function $X : [0, T] \rightarrow (0, 1]$ satisfies the prescribed terminal condition. On the other hand, differentiating (26) and comparing the result with (23) we obtain

$$-dX_t = -\frac{1}{4}X_t \|\xi_t\|^2 dt = \left(\hat{g}[t, X_t, Y_t]dt - Y_t^* dZ_t \right) \Big|_{Y_t=0}$$

and it follows that X belongs to the set of solutions to (23). The definition of the map f and (22) then show that $X = X^{\hat{\pi}}$ and it now only remains to observe that by Theorem B.4 the maximal solution to (23) is dominated by $X^{\hat{\pi}}$ to complete the proof. \square

Proof of Lemma 3. Assuming constant market coefficients and using Corollary 1 in conjunction with (2) and (3), we obtain that the equilibrium dynamics of fund value and of the investor's wealth are given by

$$dF_\tau = F_\tau \left[(r + \gamma)d\tau + \frac{2\gamma}{\|\xi\|^2} \xi^* dB_\tau \right] \quad (\text{A.3})$$

$$dW_\tau^i = W_\tau^i \left[\left(r + \frac{\|\xi\|^2}{4} \right) d\tau + \frac{1}{2} \xi^* dB_\tau \right] \quad (\text{A.4})$$

Applying Itô's lemma to (A.3) in order express the differential of $(\xi^* B)$ as a function of that of F and plugging the result back into (A.4) we obtain that the equilibrium dynamics of the investor's wealth are equivalently given by

$$\begin{aligned} \frac{dW_\tau^i}{W_\tau^i} &= r \left(1 - \frac{\|\xi\|^2}{4\gamma} \right) d\tau + \frac{\|\xi\|^2}{4\gamma} \frac{dF_\tau}{F_\tau} \\ &= \left[r \left(1 - \frac{\|\xi\|^2}{4\gamma} \right) + \frac{\gamma}{2} \right] d\tau + \frac{\|\xi\|^2}{4\gamma} d \log F_\tau \end{aligned}$$

Integrating this linear stochastic differential equation over the time interval $[t, t + \Delta]$ we obtain (35) and our proof is complete. \square

Proof of Propositions 4 & 5. The results of these propositions are obtained by differentiating the expressions defining the two measures of fund flows that we consider in this paper. We omit the details. \square

Proof of Proposition 6. Consider the modified optimization problem defined by

$$\sup_{\pi \in \mathbb{R}^n} \frac{\pi^* \sigma \xi - \gamma}{\|\sigma^* \pi\|^2} \quad \text{s.t.} \quad \pi^* \mathbf{1} = 1 \quad (\text{A.5})$$

and observe that if this problem admits a unique solution which is such that the corresponding value is strictly positive, then it must be that this solution coincides with that of (38). Solving the first order conditions associated with this problem and checking the associated Hessian constraint, we find that its unique solution is given by

$$\pi^{(\gamma)} := [\sigma^*]^{-1} \frac{\xi - \lambda^{(\gamma)} A}{\xi^* A - \lambda^{(\gamma)} \|A\|^2}$$

where $\lambda^{(\gamma)} := \gamma - \|\xi - \gamma A\| / \|A\|$ and $A := [\sigma^{-1} \mathbf{1}]$. Plugging this quantity back into the objective function, we find that the corresponding value is given by

$$\frac{1}{2} \left\{ (\xi - \gamma A)^* A + \|A\| \cdot \|\xi - \gamma A\| \right\}$$

and is non negative by application of the Cauchy-Schwartz inequality. It follows that the optimal fund portfolio is given by $\hat{\pi} = \pi^{(\gamma)}$. The remaining claims in the statement are obtained by plugging the new expression for $\hat{\pi}$ into the formulae of Sections 3 and 4, we omit the details. \square

Proof of Corollary 3. Using the results of Proposition 6 in conjunction with an argument similar to that which lead to (20), it is easily seen that the optimal fee rate is a solution to the optimization problem

$$\inf_{\gamma > 0} \exp \left[-\frac{\gamma}{2} \left(\xi - \left\{ \gamma - \frac{\|\xi - \gamma A\|}{\|A\|} \right\} A \right)^* AT \right] =: \inf_{\gamma > 0} z(\gamma).$$

Solving the first order conditions associated with this problem, we find that the unique candidate is given by $\hat{\gamma} = \|\xi\|^2 / 2(\xi^* A)$ with $z''(\hat{\gamma}) \geq 0$, we conclude that $\hat{\gamma}$ is indeed the optimal fee rate. The remaining claims in the statement follow by putting $\gamma = \hat{\gamma}$ in the formulae of Proposition 6, we omit the details. \square

Proof of Proposition 7. From Cvitanić and Karatzas (1993, Section 11) we know that the vector of optimal portfolio proportions is given by $p^\pi := \Sigma_\pi^{-1} \Lambda_\pi \xi_\pi + \mathbf{e} \hat{\nu}$ where $\hat{\nu}$ is the pointwise minimal argument of the strictly convex function

$$\|\xi_t^\pi + \nu_t [\Lambda_t^\pi]^* [\Sigma_t^\pi]^{-1} \mathbf{e}\|^2$$

over the set non negative scalar processes ν . Solving this convex programming problem we find that the unique minimizer is given by the adapted process

$$\hat{\nu}_t \triangleq \|\Lambda_t^\pi [\Sigma_t^\pi]^{-1} \mathbf{e}\|^{-2} \left(\mathbf{e}^* [\Sigma_t^\pi]^{-1} \Lambda_t^\pi \xi_t^\pi \right)^- = \frac{(\mathbf{e}^* C_t^\pi)^-}{\mathbf{e}^* [\Sigma_t^\pi]^{-1} \mathbf{e}}$$

Plugging this back into the expression for p^π and using the definition of the variance covariance matrix in conjunction with that of the process A^π we find that the investor's best response vector of portfolio proportions is given by

$$p_t^\pi = C_t^\pi + A_t^\pi (\mathbf{e}^* C_t^\pi)^-.$$

The result now follows from the definition of the vector of portfolio proportions and the fact that the first component of the investor's best response trading strategy is given by $\phi^\pi = [\psi^{i\pi}]^1 = \mathbf{e}^* \psi^\pi$. \square

Proof of Lemma 4. As is easily seen from (46), the process M^π defined in the statement is the stochastic exponential of the process

$$N_t^\pi := \int_0^t [p_\tau^\pi]^* \Lambda_\tau^\pi dZ_\tau.$$

and is a positive local martingale under the risk neutral probability measure. In order to show that M_π is a uniformly integrable martingale, it is therefore sufficient to establish that it satisfies the Novikov condition

$$E_Q \left[\exp \left(\int_0^T \frac{\|\Lambda_\tau^\pi]^* p_\tau^\pi\|^2}{2} d\tau \right) \right] < \infty. \quad (\text{A.6})$$

Using the result of Proposition 7 in conjunction with the definition of the vector valued processes (C^π, A^π) we obtain that the inequality

$$\|[\Lambda_\tau^\pi]^* p_\tau^\pi\|^2 \leq \xi_\tau^* \left([\Lambda_\tau^\pi]^* [\Sigma_\tau^\pi]^{-1} \Lambda_\tau^\pi \right) \xi_\tau + \left(\frac{[\pi_\tau^* \sigma_\tau \Omega_\tau \xi_\tau - \gamma]^+}{\sqrt{\pi_\tau^* \sigma_\tau \Omega_\tau \sigma_\tau^* \pi_\tau}} \right)^2 =: c_\tau^\pi$$

holds for all admissible fund portfolios. Using the standard formulae for the inversion of block matrices, as found for example in Johnston (1972,p.93–94), in order to compute the matrix product $[\Lambda^\pi]^* [\Sigma^\pi]^{-1} \Lambda^\pi$ we then get that the process c_τ^π is independent from the choice of π and given by

$$c_\tau^\pi = \xi_\tau^* \Omega_\tau^0 \xi_\tau = \|\xi_\tau\|^2 - \xi_\tau^* \bar{\sigma}_\tau^* [\bar{\sigma}_\tau \bar{\sigma}_\tau^*]^{-1} \bar{\sigma}_\tau \xi_\tau.$$

The validity of the Novikov condition (A.6) now follows from the boundedness of the risk premium process ξ and of the matrix valued volatility process $\bar{\sigma}$, we omit the details. \square

Proof of Proposition 8. Using the standard formulae for the inversion of block matrices, as found for example in Johnston (1972,p.93–94), in conjunction with the definition of the process C^π we easily obtain that

$$\max_{\pi \in \mathbb{R}^n} [\mathbf{e}^* C_t^\pi]^+ = \max_{\pi \in \mathbb{R}^n} \frac{[\pi^* \sigma_t \Omega_t \xi - \gamma]^+}{\pi^* \sigma_t \Omega_t \sigma_t^* \pi} =: \max_{\pi \in \mathbb{R}^n} k(\pi)$$

Ignoring the plus sign and differentiating k with respect to the fund portfolio, we obtain that the first order conditions are given by

$$\frac{\sigma_t \Omega_t \xi_t [\pi^* \sigma_t \Omega_t \sigma_t^* \pi] - 2 \sigma_t \Omega_t \sigma_t^* \pi [\pi^* \sigma_t \Omega_t \xi - \gamma]}{[\pi^* \sigma_t \Omega_t \sigma_t^* \pi]^2} = \mathbf{0}_n. \quad (\text{A.7})$$

Multiplying both sides by π^* and solving the resulting equation we find that every candidate optimizer must satisfy $\pi^* \sigma_t \Omega_t \xi = 2\gamma$. Plugging this back into (A.7) we now have that the first order conditions are equivalently given by

$$\Omega_t \left[\xi_t - \frac{2\gamma \sigma_t^* \pi}{\pi^* \sigma_t \Omega_t \sigma_t^* \pi} \right] = \mathbf{0}_n.$$

Solving this system of equations we then obtain that the set of non zero vectors in \mathcal{S}_t coincides with the set of solutions to (A.7). Observing that the value $k(\pi)$ is strictly positive and independent of the choice of $\pi \in \mathcal{S}_t \setminus \{0\}$ we conclude that every such π is a candidate optimizer and it now only remains to check the associated Hessian constraint the complete the proof. \square

Proof of Corollary 4. From Proposition 8 we have that at an arbitrary time t the optimal fund portfolio is any vector of the form

$$\pi_t = [\sigma_t^*]^{-1} \left[L_t + \frac{2\gamma \xi_t}{\xi_t^* \Omega_t \xi_t} \right]$$

for some $L_t \in \Omega_t^\perp$. Thus, all there is to prove is that for the candidate optimal fund portfolio process $\hat{\pi}$ defined in the statement we have

$$\Omega_t \left[\sigma_t^* \hat{\pi}_t - \frac{2\gamma \xi_t}{\xi_t^* \Omega_t \xi_t} \right] = \mathbf{0}_n.$$

This easily follows from the definition of the matrix Ω_t and the fact that by assumption the rows of the volatility matrix σ_t are linearly independent. We omit the details. \square

Proof of Lemma 5. Assume that we extend the investor's opportunity set from m to $m+1$ stocks. Let $v^* \in \mathbb{R}^n$ be the volatility vector of the additional stock, denote by $\tilde{\sigma}$ the volatility matrix of the $m+1$ traded stocks and set

$$\tilde{\Omega}_t^0 := \tilde{\sigma}_t^* [\tilde{\sigma}_t \tilde{\sigma}_t^*]^{-1} \tilde{\sigma}_t$$

Using the definition of the function u in conjunction with the standard formulae for the inversion of block matrices, we find that

$$\begin{aligned} u(m+1) &= \int_0^T \left[\frac{3\xi_\tau^* \tilde{\Omega}_\tau^0 \xi_\tau}{8} \right] d\tau = \int_0^T \left[\frac{3\xi_\tau^* \Omega_\tau^0 \xi_\tau}{8} + \frac{(\xi_\tau^* \Omega_\tau v_\tau^*)^2}{v_\tau \Omega_\tau v_\tau^*} \right] d\tau \\ &= u(m) + \int_0^T \left[\frac{(\xi_\tau^* \Omega_\tau v_\tau^*)^2}{v_\tau \Omega_\tau v_\tau^*} \right] d\tau. \end{aligned}$$

Observing that the matrix Ω_t is positive semidefinite by construction we easily deduce from the above equation that $u(m) \leq u(m+1)$ and the desired result now follows from a standard induction argument. \square

B Backward Stochastic Differential Equations

In this appendix, we introduce the notion of adapted solution to a Backward Stochastic Differential Equation (BSDE) driven by a n -dimensional Brownian motion and briefly review the different results that were used in the paper. In order to simplify the presentation, and since results for the vector valued case will not be used in the text, we restrict ourselves to the scalar valued case.

B.1 An Existence Result

Throughout this appendix, we work on a probability space (Ω, \mathcal{F}, P) similar to that of Section 2. On this basis, a solution to the BSDE with parameters (ξ, f) is a pair (X, Y) of adapted processes with value in $\mathbb{R} \times \mathbb{R}^n$ such that

$$-dX_t = f(t, X_t, Y_t) dt - Y_t^* dB_t, \quad X_T = \xi. \quad (\text{B.1})$$

The progressively measurable mapping f is referred to as the *driver* and the measurable random variable ξ is called the *terminal condition*.

Before giving sufficient conditions for the existence and uniqueness of the solution to (B.1) let us first fix some notation. For every $p \in [1, \infty)$ we denote by \mathbb{H}_d^p the set of adapted, \mathbb{R}^d -valued processes h with

$$\|h\|_p := E \left[\int_0^T \|h_\tau\|^p d\tau \right]^{1/p} < \infty$$

and let \mathbb{H}_d^∞ denote the set of progressively measurable, \mathbb{R}^d -valued processes whose Euclidean norm is essentially bounded. Using these notations, we now define the notion of *standard parameters* for the BSDE (B.1).

Definition B.1 *The data (ξ, f) are said to be standard parameters if (i) the progressively measurable process $f(\cdot, 0, 0)$ belongs to the space \mathbb{H}_1^2 , (ii) we have*

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq K (|x_1 - x_2| + |y_1 - y_2|)$$

almost surely for all $((t, x_i, y_i))_{i=1}^2$ and some strictly positive constant K and (iii) the terminal condition is a square integrable random variable.

Theorem B.1 *Given a pair of standard parameters as above, there exists a unique adapted solution (X, Y) to the BSDE (B.1) in the space $\mathbb{H}_1^2 \times \mathbb{H}_n^2$.*

Proof. This result was originally proved by Pardoux and Peng (1990). Various extensions and refinements of this existence result can be found in the research monograph edited by El Karoui and Mazliak (1997). \square

B.2 The Linear Case

In this section we specialize the above existence result to the case where the driver is linear in (x, y) . Let (γ, φ, β) denote a triple of progressively measurable bounded processes with values in the product space $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$ and consider the BSDE (B.1) with driver given by

$$f(t, x, y) := \gamma_t - \varphi_t x - \beta_t^* y.$$

Such equations are referred to as *linear* backward stochastic differential equations (LBSDEs) and admit an explicit solution as we now demonstrate.

Proposition B.1 *If ξ is square integrable, then the unique solution to the linear BSDE with terminal condition ξ and driver f is explicitly given by*

$$X_t = E \left[\frac{H_T \xi}{H_t} + \int_t^T \frac{\gamma_\tau H_\tau}{H_t} d\tau \middle| \mathcal{F}_t \right]$$

where the adjoint process H is defined as the unique solution to the linear forward stochastic differential equation

$$dH_t = -H_t [\varphi_t dt + \beta_t^* dB_t], \quad H_0 = 1.$$

In particular, if the random variable ξ and the process γ are non negative, then X is a non negative process and if moreover $X_0 = 0$ then we have $\xi = 0$ almost surely as well $X = \gamma = 0$ almost everywhere.

Proof. See the proof of Proposition 2.4 in Chapter 1 of the research monograph edited by El Karoui and Mazliak (1997). \square

As an immediate consequence of the above proposition we obtain the celebrated comparison theorem which was first stated by Peng (1992). This result holds only in the one dimensional case and plays a role similar to that of the maximum principle in the theory of partial differential equations.

Theorem B.2 *Let $((\xi_k, f_k))_{k=1}^2$ be standard parameters and assume that*

$$(\xi_1, f_1(t, x, y)) \leq (\xi_2, f_2(t, x, y)), \quad \forall (t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}^n$$

holds almost surely. Then the trajectory of the corresponding solutions are such that the inequality $X_1 \leq X_2$ holds almost everywhere

Proof. See the proof of Theorem 2.5 in Chapter 1 of the research monograph edited by El Karoui and Mazliak (1997). \square

B.3 Optimization of a Family of BSDEs

Let us now suppose that we are given a set \mathcal{A} of progressively measurable processes α and a family $\{(\xi_\alpha, f_\alpha)\}$ of standard parameters. Since it gives very simple conditions for the solution of a backward equation to dominate that of another, the comparison Theorem B.2 makes BSDEs a practical tool for the study of non Markovian stochastic control problems.

Corollary B.1 *If there exists a process $\hat{\alpha} \in \mathcal{A}$ such that*

$$(\xi_\alpha, f_\alpha(t, x, y)) \leq (\xi_{\hat{\alpha}}, f_{\hat{\alpha}}(t, x, y)), \quad \forall (t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}^n$$

holds almost surely for all $\alpha \in \mathcal{A}$ and if the parameters associated with the process $\hat{\alpha}$ are standard, then we have $X_\alpha \leq X_{\hat{\alpha}}$ almost everywhere for all $\alpha \in \mathcal{A}$.

The difficulty in applying the above result to solve a particular stochastic control problem is that even if the set \mathcal{A} is well behaved enough for the data (ξ_α, f_α) to be standard parameters, the *optimal control* might not possess this property and the corresponding BSDE might not even admit an adapted solution. In particular, the above result does not apply to the stochastic control problem studied in Section 3.2.2 since the driver under consideration fails to satisfy the requirements of Definition B.1.

The next theorem extends the results of the previous subsections by allowing for a driver with superlinear growth in x and quadratic growth in y . Note that because we lessen the

conditions imposed on the driver, we are forced to strengthen our integrability assumption on the terminal condition and that uniqueness of the solution is lost. In the following statement, we say that the pair (X, Y) of adapted processes is a *maximal solution* to the BSDE if its trajectory dominates that of any other solution.

Theorem B.3 *Assume that ξ is bounded, that f is continuous in (t, x, y) and that there exist non negative constants (C, D, E) such that*

$$|f(t, x, y)| \leq C + D|x| + E\|y\|^2, \quad \forall (t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}^n$$

holds almost surely. Then the BSDE with parameters (ξ, f) admits a unique maximal solution in the product space $\mathbb{H}_1^\infty \times \mathbb{H}_n^2$.

Proof. For the case of a driver bounded in x and with quadratic growth in y , this result was proved by Kobylanski (1997) who also finds sufficient conditions for uniqueness of the solution. The version presented here is from Lepeltier and San Martin (1998) to which the reader is referred. \square

Our last result in this appendix is also taken from the paper by Lepeltier and San Martin who obtain it as a by product of the method they use to prove Theorem B.3. It establishes a comparison result for the solution to the BSDE and was used in the resolution the stochastic control problem associated with the determination of the equilibrium fund portfolio process in Section 3.2.2.

Theorem B.4 *Let h denote a progressively measurable mapping, θ be a bounded measurable random variable and assume that (ξ, f) satisfy the conditions of Theorem B.3. If $(\xi, f) \leq (\theta, h)$ holds almost surely then any bounded solution to the backward equation associated with the pair (θ, h) dominates the maximal solution to the backward equation associated with the pair (ξ, f) .*

Proof. See the proof of Corollary 2 in Lepeltier and San Martin (1998). \square

Example B.1 In some special cases, the BSDEs covered by Theorem B.3 admit explicit solutions. For example, the equation with driver given by $\beta\|y\|^2$ for some constant $\beta \neq 0$ and bounded terminal condition ξ admits a unique solution whose trajectory is given by

$$X_t = \frac{1}{2\beta} \log \left(E \left[\exp 2\beta\xi \middle| \mathcal{F}_t \right] \right).$$

For another example, let θ denote an arbitrary bounded, predictable process with values in \mathbb{R}^n , define an equivalent probability measure by setting

$$P_\theta(A) := E \left[1_A \exp \left(- \int_0^T \frac{1}{8} \|\theta_\tau\|^2 d\tau + \int_0^T \frac{1}{2} \theta_\tau^* dB_t \right) \right] \quad (\text{B.2})$$

and consider the backward equation with constant terminal condition $\xi = 1$ and driver given by $-(x/4) \|-y/x + \theta_t\|^2$. Assuming that its maximal solution is strictly positive and applying Itô's lemma to the square root of this maximal solution, we obtain a candidate solution in the form

$$X_t^{\frac{1}{2}} = E_\theta \left[\exp \left(- \int_t^T \frac{1}{8} \|\theta_\tau\|^2 d\tau \right) \middle| \mathcal{F}_t \right]$$

where E_θ denotes the expectation operator under the probability measure of (B.2). It is now straightforward to check that the process X is strictly positive and constitutes the unique bounded solution to the BSDE under consideration. Other examples of explicitly solvable quadratic equations may be found in El Karoui and Rouge (2000).

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