Market Diversity and The Distribution of Capital in Equity Markets

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Outline

1. Capital distribution curves

2. Stability and the size effect

3. Diversity-weighted portfolios

4. The distributional component of return

References

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2. Fernholz, R. *Stochastic Portfolio Theory.* Springer (2002).

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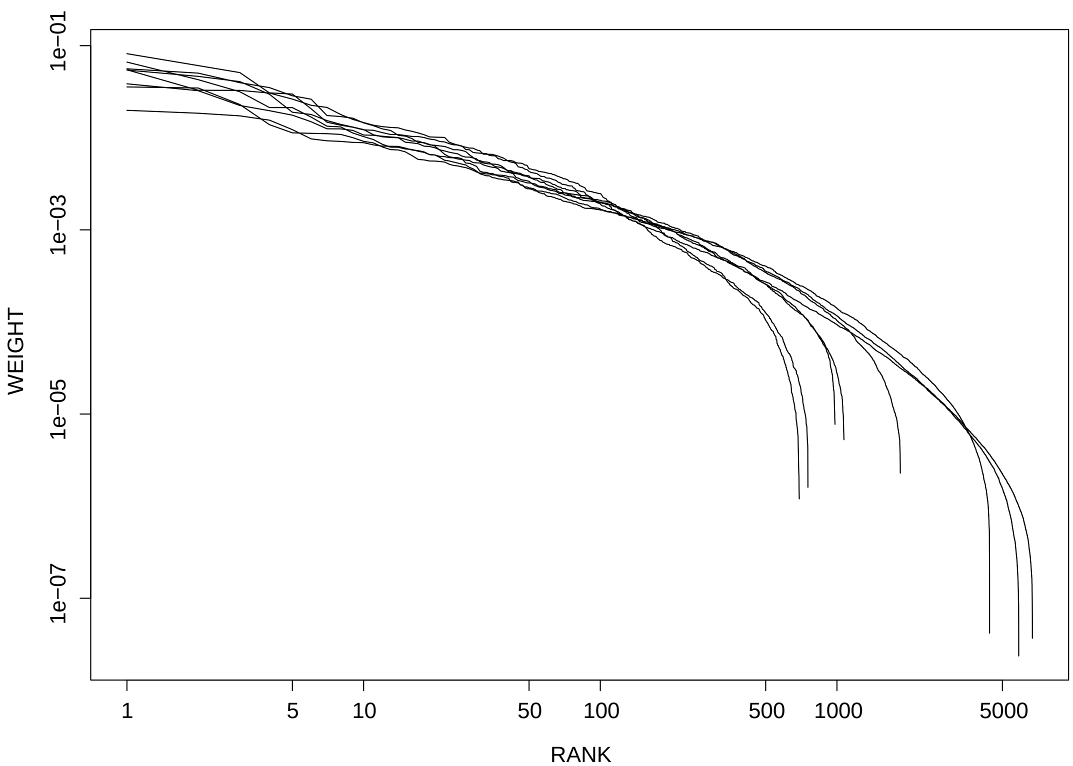
Capital distribution curves

For an equity market, the capital distribution curve is the graph of ranked capitalisation weight v. rank (with 1 representing the largest-cap stock). Plot on log-log axes to reveal the most information.

For US equity markets (combined NYSE, AMEX, NASDAQ), sampled every 10 years from 1929 through 1999, the curves look similar:



Rates of return and growth rates

Standard SDE for a stock price:

$$\frac{dX(t)}{X(t)} = \alpha(t)\,dt + \sigma(t)\,dW(t)$$

Logarithmic version:

$$d(\log(X(t))) = \gamma(t)\,dt + \sigma(t)\,dW(t)$$

Itô's rule gives

$$\gamma(t) = \alpha(t) - \frac{1}{2}\sigma^2(t).$$

Logarithmic representation is better because

$$\lim_{T\to\infty} \frac{1}{T}\log(X(t)) = \frac{1}{T}\int_0^T \gamma(t)\,dt$$

by the strong law of large numbers. We call $\gamma(t)$ the *growth rate* (or geometric rate of return).

Stability of capital distribution

Theorem (Fernholz): if all stocks have the same $\gamma(t)$ and bounded covariance, then the market collapses to a single stock:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mu_{(1)}(t)\,dt = 1$$

where $\mu_{(1)}(t)$ is the greatest market weight.

So what stabilizes a market? Still an open question, but progress has been made. Likely due to a combination of three factors, each of which could be enough on its own:

1. Infusion of new stocks into the market

2. Small-cap stocks may have higher growth rates than large-cap stocks

3. Small-cap stocks may have higher volatility than large-cap stocks

Stability and the size effect

Suppose the capital distribution is very stable. For example, suppose that

$$\text{cap(top 100 stocks)} = \text{cap(next 900 stocks)}$$

at all times. Then the portfolio **A** of the top 100 stocks underperforms the portfolio **B** of the next 900 stocks, since **A** always sells after a relative downward movement and buys after a relative upward movement. The opposite is true for **B**.

In reality, the above equation is not true. The movement in relative capital affects the situation.

Measuring the size effect

Let $R(t)$ be the ratio of (combined cap of stocks ranked 101-1000) to (combined cap of stocks ranked 1-100) at time t.

Let $Z_A(t)$ and $Z_B(t)$ be the portfolio values of **A** and **B** respectively (assume the same initial value).

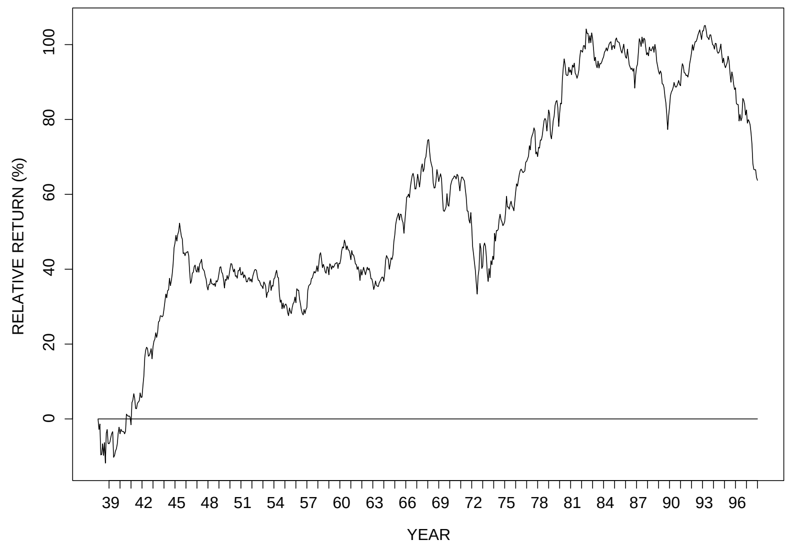
Then

$$d\log\left(\frac{Z_B(t)}{Z_A(t)}\right) = d\log(R(t)) + d\Theta(t)$$

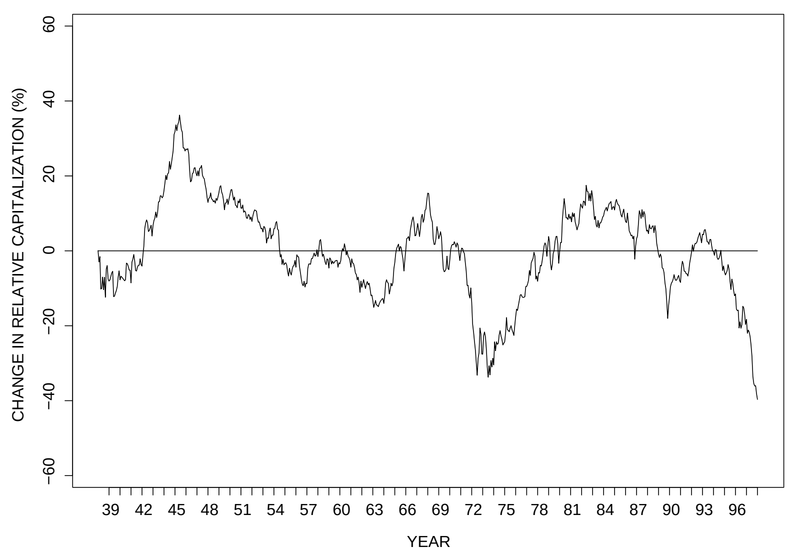
where $\Theta(t)$ is a monotonically increasing drift process whose formula involves Brownian local time.

So, if $R(t)$ is constant, then **B** wins. If $R(t)$ decreases for a period of time, then $\log(R(t))$ can decline more than $\Theta(t)$ gains, and **A** can win over that period of time. If the market is stable in the long term, **B** wins.

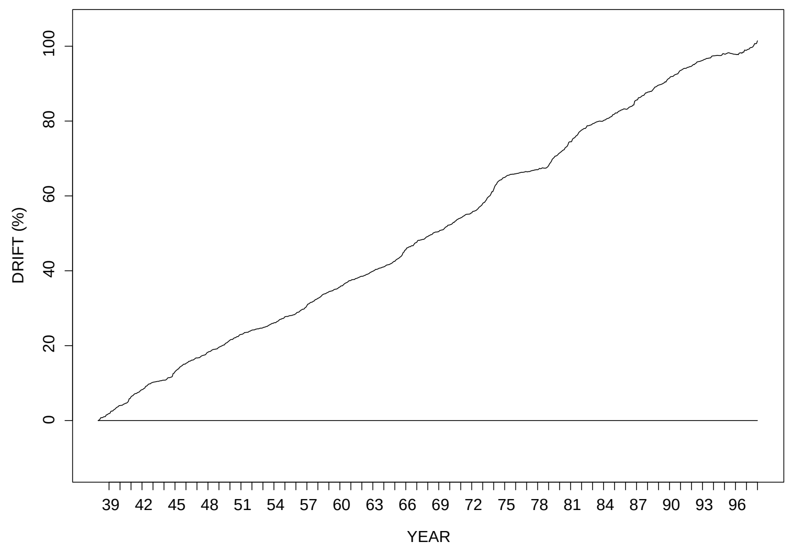
Simulating the portfolios, here is $\log(Z_B/Z_A)$:



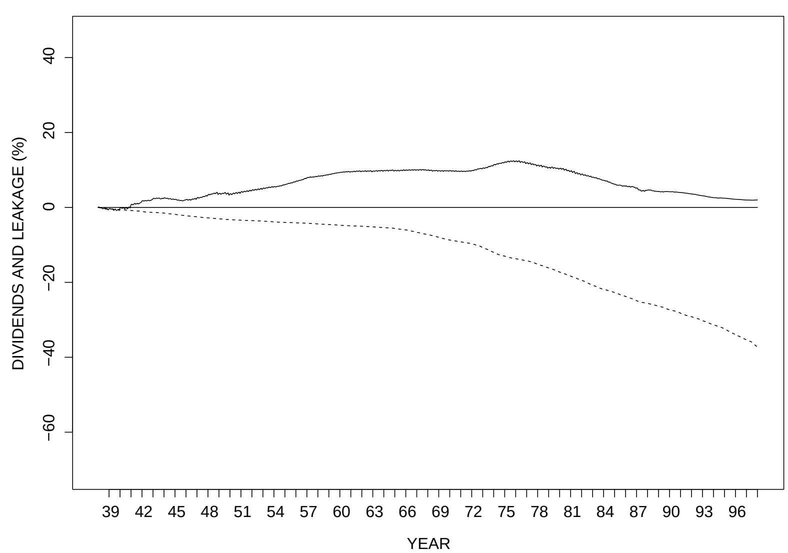
And here is $\log(R)$:



The difference is the drift Θ (incl. leakage)



There are dividends (solid line) and **B** has leakage (stocks fall out of the top 1000). The overall drift is still positive.



Diversity

The quantity $R(t)$ measures the small oscillations in the capital distribution curve. A similar measure is given by *diversity*:

$$D_p(t) = \left(\sum_{i=1}^{n} (\mu_i(t))^p \right)^{1/p}$$

where $0 < p < 1$ is fixed and $\mu_i(t)$ are the market cap weights at time t.

Change in diversity for $p = 1/2$ and $n = 1000$:

Diversity-weighted portfolio

Exploit via *diversity weighting*:

$$\pi_i(t) = \frac{(\mu_i(t))^p}{\sum_{j=1}^n (\mu_j(t))^p}$$

The resulting portfolio **D** is a reweighting of the index which respects rank (e.g. #1 stock in index is also #1 stock in **D**).

Let $Z_D(t)$ and $Z_I(t)$ be the portfolio values of **D** and the index respectively (assume the same initial value). Then

$$d\log\left(\frac{Z_D(t)}{Z_I(t)}\right) = d\log(D_p(t)) + d\Theta(t)$$

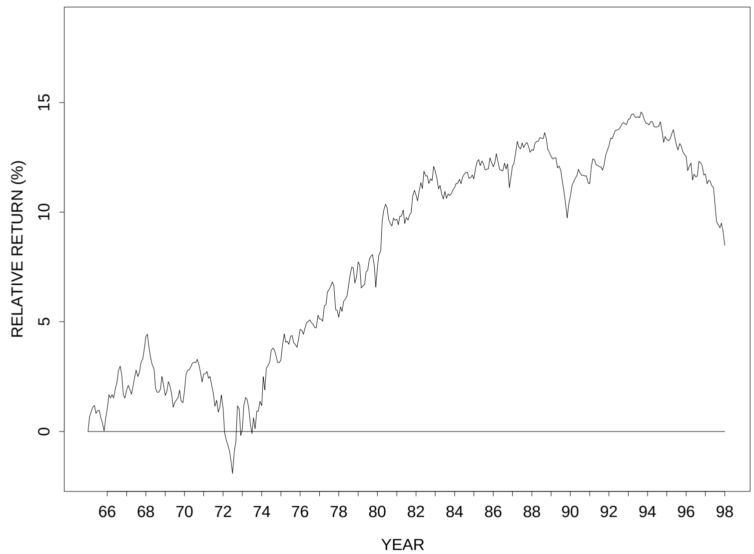
where $\Theta(t)$ is monotonically increasing:

$$\Theta(t) = (1-p)\int_0^t \gamma_\pi^\star(s)\,ds$$

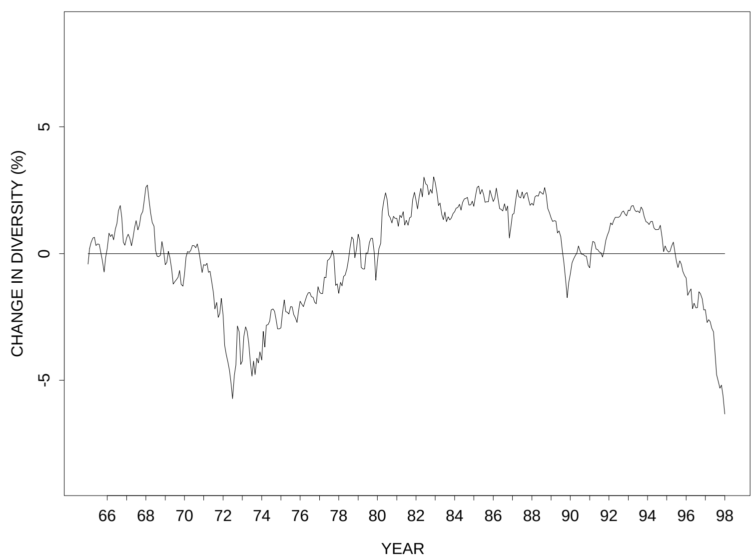
and the excess growth $\gamma_\pi^\star(t)$ is

$$\gamma_\pi^\star(t) = \frac{1}{2}\left(\sum_{i=1}^n \pi_i(t)\sigma_i^2(t) - \sum_{i,j=1}^n \pi_i(t)\pi_j(t)\sigma_{ij}(t)\right).$$

Simulating with I=S&P500, here is $\log(Z_D/Z_I)$:



And here is $\log(D_p)$:



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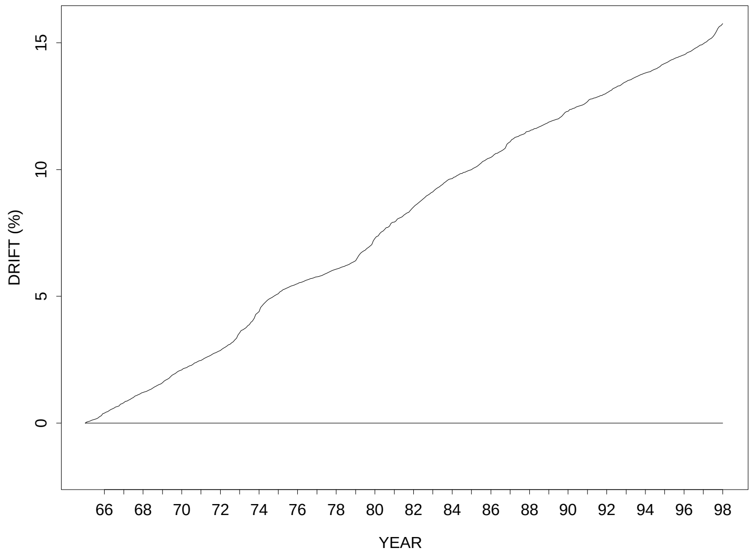
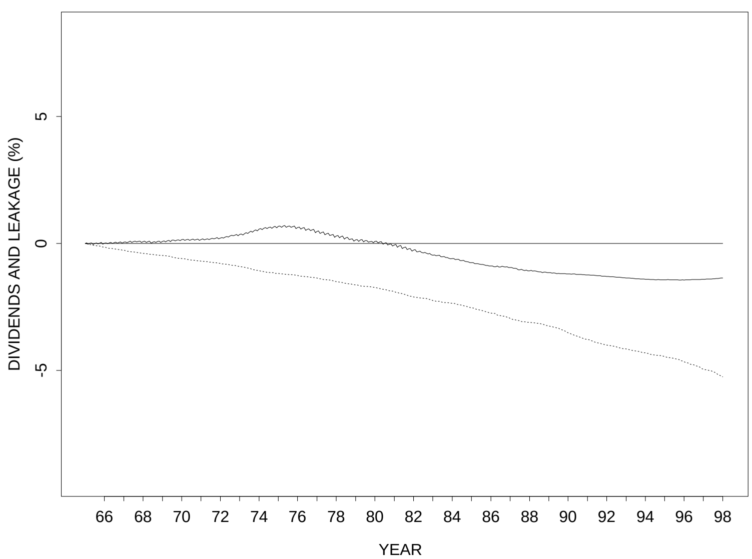


There are dividends (solid line) and **B** has leakage (stocks falling out of the S&P hurt **D** more than the index). Overall drift is still positive.



Distributional component of return

In summary, the relative log return of the diversity-weighted portfolio has two pieces:

1. change in diversity, which comes from movement of the capital distribution curve; and

2. the drift, which arises from stocks changing rank (better to overweight small stocks if the capital distribution curve is constant).

We seek to decompose the relative log return of *any* portfolio into two similar pieces:

1. the distributional component, measuring the impact of changes in the capital distribution curve on the portfolio; and

2. the residual, measuring the effect of changes in rank on the portfolio.

Distributional component of return (ctd.)

Start with equal amounts of portfolio π and index μ. Relative log return over the (discrete) time period from t_0 to t_1 is

$$\log\left(\frac{Z_\pi(t_1)}{Z_\mu(t_1)}\right) = \log\left(\sum_{i=1}^{n} \pi_i(t_0)\frac{\mu_i(t_1)}{\mu_i(t_0)}\right).$$

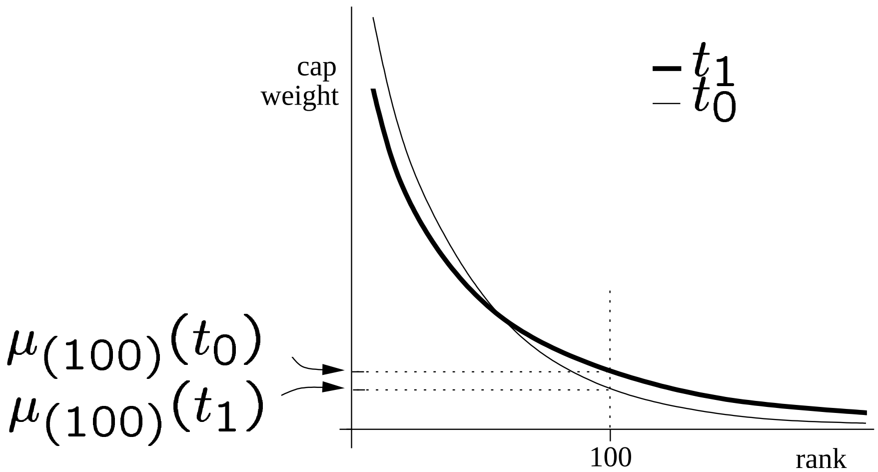
Now let $\mu_{(i)}(t)$ be the index weight of the ith-ranked stock at time t, and $\pi_{(i)}(t)$ be the portfolio weight of the same stock. Define

$$DC_{t_0 \to t_1} = \log\left(\sum_{i=1}^{n} \pi_{(i)}(t_0)\frac{\mu_{(i)}(t_1)}{\mu_{(i)}(t_0)}\right).$$

to be the distributional component of relative log return.

If the curve doesn't move, this is zero; whereas if the ranks don't change, this is the entire relative log return.

Over the time period from t_0 to t_1, suppose (for example) that the curve changes:



Return due to rank 100 is the ranked weight ratio $\mu_{(100)}(t_1)/\mu_{(100)}(t_0)$. The distributional component is just the weighted average across ranks, and residual is what's left:

$$DC_{t_0 \to t_1} = \log\left(\sum_{i=1}^{n} \pi_{(i)}(t_0)\frac{\mu_{(i)}(t_1)}{\mu_{(i)}(t_0)}\right).$$

$$RES_{t_0 \to t_1} = \log\left(\frac{Z_\pi(t_1)/Z_\pi(t_0)}{Z_\mu(t_1)/Z_\mu(t_0)}\right) - DC_{t_0 \to t_1}.$$

The DC gives an excellent measure of the impact of size on the portfolio, without any assumptions (unlike in the case of regression).