The Connection Between BSDE and PDE and Numerical Methods to Solve BSDE

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Outline

- SDE , BSDE and PDE, an Example
- Forward-Backward Stochastic Differential Equations
- BSDE and PDE: Nonlinear Feynman-Kac Formula
- Numerical Methods to Solve BSDE

1 PDE and SDE: An Example

- The expectation of an Ito process can be obtained by solving the associated PDE. On the other hand, for certain PDEs, we can express its solution by an expectation of an Ito process.
- Ito's formula and the Dynkin's formula
- Feynman-Kac formula

- **Example:** Consider the backward parabolic PDE:

$$\begin{aligned} \partial_t u(t,x) + (\mathcal{L}u)(t,x) + c(x)u(t,x) &= 0, \quad 0 \le t \le T, x \in \mathbf{R}^d, \\ u(T,x) &= g(x), \end{aligned}$$

where $\mathcal{L}$ is defined by

$$\mathcal{L} = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + b\frac{\partial}{\partial x}. \tag{1}$$

The PDE has a probability solution.

More particular, let X_t be the solution of the following SDE:

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad t \in [0,T], \tag{2}$$

where B_t is a standard d-dimensional Brownian motion. Then, Using the Feynman-Kac formula, we can get that the solution $u(t,x)$ is given by

$$u(t,x) = \mathbf{E}^{t,x}\left[e^{\int_t^T c(X_s)ds} g(X_T)\right]. \tag{3}$$

- Now let us consider the following nonlinear equation:

$$\begin{aligned}\partial_t u(t,x) + (\mathcal{L}u)(t,x) + f(u(t,x)) &= 0, \quad 0 \le t \le T, x \in \mathbf{R}^d,\\ u(T,x) &= g(x),\end{aligned}$$

where $f : \mathbf{R} \to \mathbf{R}$ is a globally Lipschitz function. Can we get a solution as an expectation of some stochastic process?

  – Suppose we can find an adapted process $\{Y_t\}$ such that

$$Y_t = \mathbf{E}\left[g(X_T) - \int_t^T f(Y_s)ds \middle| \mathcal{F}_t\right], \quad 0 \le t \le T, \tag{4}$$

and define $u(t, X_t) = Y_t$. Then we have

$$u(t, X_t) = \mathbf{E}\left[g(X_T) - \int_t^T f(u(s, X_s))ds \middle| \mathcal{F}_t\right], \quad 0 \le t \le T, \tag{5}$$

and the solution is

$$u(t,x) = \mathbf{E}^{t,x}\left[g(X_T) - \int_t^T f(Y_s)ds\right]. \tag{6}$$

– We need to find the SDE for Y_t. Actually, Y_t is a solution of a BSDE:

$$\begin{aligned} dY_s^{t,x} &= f(Y_s^{t,x})ds + Z_s dB_s, \quad s \in [t,T], \\ Y_T^{t,x} &= g(X_T^{t,x}). \end{aligned}$$

where $X_s^{t,x}$ is a solution of a normal SDE:

$$\begin{aligned} dX_s^{t,x} &= b(X_s^{t,x})ds + \sigma(X_s^{t,x})dB_s, \quad s \in [t,T], \\ X_t^{t,x} &= x. \end{aligned}$$

– Consider the more general semilinear parabolic PDE:

$$\begin{aligned} \partial_t u(t,x) + (\mathcal{L}u)(t,x) + f(t,x,u(t,x),Du(t,x)) &= 0, \quad 0 \le t \le T, x \in \mathbf{R}^d, \\ u(T,x) &= g(x). \end{aligned}$$

We need to consider the more general BSDE:

$$\begin{aligned} dY_s^{t,x} &= f(t, X_s^{t,x}, Y_s^{t,x}, Z_s)ds + Z_s dB_s, \quad s \in [t,T], \\ Y_T^{t,x} &= g(X_T^{t,x}). \end{aligned}$$

2 Forward-Backward Stochastic Differential Equations

- Consider a classical (forward) stochastic differential equation (SDE) and related it to a backward stochastic differential equation (BSDE). The related solution of BSDE is a type of nonlinear PDE of parabolic type.

- Forward stochastic differential equation with initial condition $(t, \zeta) \in [0, T] \times L^2(\mathcal{F}_t, P; \mathbf{R}^n)$:

$$dX_s^{t,\zeta} = b(\omega, s, X_s^{t,\zeta})ds + \sigma(\omega, s, X_s^{t,\zeta})dB_s, \quad s \in [t, T], \tag{7}$$

$$X_t^{t,\zeta} = \zeta, \tag{8}$$

  where $\{B_s\}_{0 \le s \le T}$ is a d-dimensional standard Brownian motion.

- Under Lipschitz conditions, the above equation has a strong unique solution.

In addition, for each $t \in [0,T)$, $\forall \zeta, \zeta' \in L^2(\mathcal{F}_t, P; \mathbf{R}^n)$, we have

$$\mathbf{E}\left[\sup_{s\in[t,T]} |X_s^{t,\zeta} - X_s^{t,\zeta'}|^2 | \mathcal{F}_t\right] \le C_0|\zeta - \zeta'|^2, \quad \text{a.s.,} \tag{9}$$

and for each $p \ge 2$, we have

$$\mathbf{E}\left[\sup_{s\in[t,T]} |X_s^{t,\zeta}|^p\right] \le C_p(1 + |\zeta|^p), \quad \forall \zeta \in L^p(\mathcal{F}_t, P; \mathbf{R}^n), \text{a.s..} \tag{10}$$

- Remark: A special case is that the initial condition is a deterministic vector in $\mathbf{R}^n$:

$$\begin{aligned} dX_s^{t,x} &= b(\omega, s, X_s^{t,x})ds + \sigma(\omega, s, X_s^{t,x})dB_s, \quad s \in [t,T], \\ X_t^{t,x} &= x, \end{aligned}$$

where $x \in \mathbf{R}^n$. We can get similar results.

- Now we consider the following BSDE:

$$-dY_s^{t,\zeta} = f(s, X_s^{t,\zeta}, Y_s^{t,\zeta}, Z_s^{t,\zeta})ds - Z_s^{t,\zeta}dB_s, \quad s \in [t,T], \tag{11}$$

$$Y_T^{t,\zeta} = \Phi(X_T^{t,\zeta}), \tag{12}$$

where $f(t,x,y,z)$ and $\Phi(x)$ are $\mathbf{R}^m$ valued functions and $(x,y,z) \in \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^{m\times d}$.

- The above equation has a strong solution if f, Ψ are bounded and satisfy

$$\begin{aligned} &|\Phi(x) - \Phi(x')| + |f(t,x,y,z) - f(t,x',y',z')| \\ &\quad \leq C(|x-x'| + |y-y'| + |z-z'|), \\ &|f(t,x,0,0)| + |\Phi(x)| \leq C(1+|x|). \end{aligned}$$

- Moreover, the solution also satisfies

$$\begin{aligned} |Y_t^{t,\zeta} - Y_t^{t,\zeta'}| &\leq C_0|\zeta - \zeta'|^2, \\ |Y_t^{t,\zeta}| &\leq C_0(1+|\zeta|). \end{aligned}$$

- Now, define a function of (t,x) by

$$u(t,x) \equiv Y_s^{t,x}|_{s=t}, \quad x \in \mathbf{R}^n. \tag{13}$$

- Properties of $u(t,x)$:

$$\begin{aligned} |u(t,x) - u(t,x')| &\leq C_0|x-x'|^2, \\ |u(t,x)| &\leq C_0(1+|x|). \end{aligned}$$

- **Remark:** For each $x \in \mathbf{R}^n$, $u(\cdot,x)$ is a $\mathcal{F}_t$-adapted process. In other words, u is a random function. However, if for each (t,x,y,z), $b(t,x)$, $\sigma(t,x)$, $\Phi(x)$, $f(t,x,y,z)$ are deterministic functions, then $u(t,x)$ becomes a deterministic function of (t,x).

- If the initial value is a random variable instead of a deterministic vector, for each $\zeta \in L^2(\omega, \mathcal{F}_t, P, \mathbf{R}^n)$, then we have

$$u(t,\zeta) = Y_t^{t,\zeta}. \tag{14}$$

3 Nonlinear Feynman-Kac Formula and Assocaited PDEs

- Connection between BSDE and PDE can be established for simple case as well as more general cases.
- If $m = 1$ and $f(t,x,y,z) = f(t,x)$ and b, σ, Φ, f are all deterministic functions, then we have

$$u(t,x) = \mathbf{E}\left[\int_t^T f(s, X_s^{t,x})ds + \Phi(X_T^{t,x})\right] \tag{15}$$

- Let $m = 1$ and b, σ, Φ, f be all deterministic functions. If $f(t,x,y,z) = c(x)y + f_0(t,x)$, then we have the following (classical) Feynman-Kac Fromula:

$$u(t,x) = \mathbf{E}\left[\int_t^T e^{\int_t^s c(X_r^{t,x})dr} f_0(s, X_s^{t,x})ds + \Phi(X_t^{t,x})e^{\int_t^T c(X_r^{t,x})dr}\right]. \tag{16}$$

- If b, σ, Φ, f are all deterministic functions, and $f(t,x,y,z) = f(t,x)$, then $u(t,x)$ is a solution of the following PDE:

$$\partial_t u(t,x) + \mathcal{L}u(t,x) + f(t,x) = 0, \tag{17}$$

where $\mathcal{L}$ is the following second order elliptic operator:

$$\mathcal{L}\psi(t,x) = \left[\frac{1}{2}Tr(\sigma\sigma^* D^2\psi) + \langle D\psi, b\rangle\right](t,x). \tag{18}$$

- More general case: If $u(t,x) : \mathbf{R}^m \times [0,T] \to \mathbf{R}^m$ is a smooth solution of the following PDE:

$$\begin{aligned} &\partial_t u(t,x) + \mathcal{L}u(t,x) + f(t,x,u,Du\sigma) = 0, \\ &u(T,x) = \Phi(x), \end{aligned}$$

then $(u, Du\sigma)(s, X_s^{t,x})$ is the solution of the BSDE

$$-dY_s^{t,x} = f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})ds - Z_s^{t,x}dB_s, \quad s \in [t,T], \tag{19}$$

$$Y_T^{t,x} = \Phi(X_T^{t,x}). \tag{20}$$

- **Nonlinear Feynman-Kac Formula:** Let $Y_s^{t,x}$ be a solution of

$$-dY_s^{t,x} = f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})ds - Z_s^{t,x}dB_s, \quad s \in [t,T], \tag{21}$$

$$Y_T^{t,x} = \Phi(X_T^{t,x}). \tag{22}$$

and define $u(t,x) \equiv Y_t^{t,x}$. Then under certain conditions, $u(t,x)$ is a continuous function of (t,x) and it is a viscosity solution of

$$\partial_t u(t,x) + \mathcal{L}u(t,x) + f(t,x,u,Du\sigma) = 0,$$
$$u(T,x) = \Phi(x).$$

Solving Forward Parabolic Equations:

- If b, σ, f do not depend on t, then we can also find the probabilistic solution of certain parabolic equations with initial conditions using the BSDE.
- More particular, if we let

$$v(t,x) = u(T-t,x), \quad (t,x) \in [0,T] \times \mathbf{R}^d, \tag{23}$$

then v solves the system of forward parabolic PDEs:

$$\partial_t v(t,x) = \mathcal{L}v(t,x) + f(x, v, Dv\sigma), \quad t \geq 0, \; x \in \mathbf{R}^d,$$
$$v(0,x) = \Phi(x).$$

On the other hand, we have that

$$v(t,x) = Y_{T-t}^{T-t,x} = \bar{Y}_0^{t,x}, \tag{24}$$

where $\{(barY_s^{t,x}, \bar{Z}_s^{t,x})\}$ solves the BSDE

$$\begin{aligned} \bar{Y}_s^{t,x} &= f(X_s^{0,x}, \bar{Y}_s^{t,x}, \bar{Z}_s^{t,x})ds - \bar{Z}_s^{t,x} dB_s, \quad 0 \leq s \leq t, \\ \bar{Y}_t^{t,x} &= \Phi(X_t^{0,x}). \end{aligned}$$

4 Numerical Solutions of BSDEs

4.1 Euler's Scheme

- Consider the solution of the following equation:

$$\begin{aligned} dY_t &= f(t, Y_t, Z_t)dt - Z_t dB_t, \quad t \in [0,1], \\ Y_1 &= \xi. \end{aligned}$$

- Setup:
  - Let $\{\epsilon_i^n\}_{i=1,2,\ldots,n}$ be an i.i.d. Bernoulli sequence:

$$Pr(\epsilon_i^n = 1) = Pr(\epsilon_i^n = -1) = \frac{1}{2}.$$

– Define

$$B_k^n = \frac{1}{\sqrt{n}} \sum_{i=1}^{k} \epsilon_i^n,$$
$$\Delta B_{k+1}^n = B_{k+1}^n - B_k^n = \frac{1}{\sqrt{n}} \epsilon_k^n,$$
$$\mathcal{F}_k^n = \sigma\{B_k^n; 1 \le k \le n\}.$$

– Let ξ^n be $\mathcal{F}_k^n$-measurable. So there is a function Φ^n such that

$$\xi^n = \Phi^n(\epsilon_1^n, \cdots, \epsilon_n^n). \tag{25}$$

– More assumptions:

* B^n converges to B;
* ξ^n converges to ξ;
* $f(t, y, z)$ and $f^n(t, y, z)$ are $\mathcal{F}_t$ and $\mathcal{F}_t^n$ progressively measurable, respectively.
* For each (y, z) pathes, $f^n(t, y, z)$ has RCLL paths and converges to $f(t, y, z)$.

- Numerical Schemes:
  - Set

$$f^n(t,y,z) = g_k^n(y,z), \quad t \in \left[\frac{k}{n}, \frac{k+1}{n}\right), k = 0, 1, ..., n. \tag{26}$$

  - Let $y_n^n = \xi^n$, a given $\mathcal{F}_n^n$-measurable random variable.
  - Solve backwardly:

$$y_k^n = y_{k+1}^n + g_k^n(y_k^n, z_k^n)\frac{1}{n} - z_k^n \Delta B_{k+1}^n, \quad k = n-1, n-2, ..., 3, 2, 1. \tag{27}$$

In addition, we set

$$y_t^n \equiv y_k^n, \quad z_t^n \equiv z_k^n, \quad \forall t \in \left[\frac{k}{n}, \frac{k+1}{n}\right) \tag{28}$$

Now the path (y_t^n, z_t^n) are RCLL.

– Solve for z_k^n: Since $y_{k+1}^n = \Phi^n(\epsilon_1^n, ..., \epsilon_{k+1}^n)$, we can set

$$\begin{aligned} y_{k+1}^+ &= \Phi^n(\epsilon_1, ..., \epsilon_k, 1), \\ y_{k+1}^- &= \Phi^n(\epsilon_1, ..., \epsilon_k, -1). \end{aligned}$$

So y_{k+1}^+, y_{k+1}^- are $\mathcal{F}_k^n$-measurable. Now set $\epsilon_{k+1}^n = \pm 1$ in (27), and we can get

$$\begin{aligned} y_k^n &= y_{k+1}^+ + g_k^n(y_k^n, z_k^n)\frac{1}{n} - z_k^n \frac{1}{\sqrt{n}}, \\ y_k^n &= y_{k+1}^- + g_k^n(y_k^n, z_k^n)\frac{1}{n} + z_k^n \frac{1}{\sqrt{n}}. \end{aligned}$$

Now z_k^n can be uniquely solved by

$$z_k^n = \frac{y_{k+1}^+ - y_{k+1}^-}{2}. \tag{29}$$

– Solve for (y_k^n): Let g_k^n be uniformly Lipschitz with respect to (y,z). Then when n is big enough, the mapping

$$A(y) \equiv y - g_k^n(y, z_k^n)\frac{1}{n} \tag{30}$$

is strictly monotonic function of y with $A(y) \to \infty$ as $y \to \infty$. Therefore, the solution y_k^n of (27) exists and is unique once $y_k + 1^n$ is known.

– Convergence results:

$$(y^n, \int_0^1 z_s^n dB_s^n) \to (y, \int_0^1 z_s dB_s), \quad \text{as} n \to \infty. \tag{31}$$

under the super-L^2 norm

$$\| Y \| = \left(\mathbf{E}\left[\sup_{0\le t\le 1} |Y_t|^2\right]\right)^{\frac{1}{2}}. \tag{32}$$

4.2 An Alternative Numerical Method

– Consider the following BSDE:

$$Y_t = \xi + \int_t^1 f(s, Y_s)ds - \int_t^1 Z_s dB_s. \tag{33}$$

– Discrete version:

$$y_i^n = \Phi(\epsilon_1, ..., \epsilon_n) + \frac{1}{n}\sum_{j=i}^{n} f(t_j, y_j^n) - \sum_{j=i}^{n-1} z_j^n \Delta B_j^n. \tag{34}$$

– Solving the above problem for y_i^n is equivalent to finding a solution for

$$y_i^n = \mathbf{E}\left[y_{i+1}^n + \frac{1}{n} f(t_i, y_i^n) \middle| \mathcal{F}_i^n \right] \tag{35}$$

or

$$y_i^n - \frac{1}{n} f(t_i, y_i^n) = \mathbf{E}\left[y_{i+1}^n \middle| \mathcal{F}_i^n \right]. \tag{36}$$

– When y_{i+1}^n is determined, y_i^n can be solved by a fixed point technique:

$$\begin{aligned} X^0 &= \mathbf{E}\left[y_{i+1}^n \middle| \mathcal{F}_i^n \right], \\ X^{k+1} &= X^0 + \frac{1}{n} f(t_i, X^k). \end{aligned}$$

– Easy to implement and does not need to assume the discretized filtrations 'converge' to the original Brownian filtration.

– Ma, Protter, Martin and Torres, 2002.