Parameter Uncertainty, application to Portfolio Choice

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Definition

Estimation Risk (**ER**) = uncertainty about the true values of some model parameters

Comments:

i) Samples are finite $\Rightarrow$ estimates are different from true parameters.

ii) ER exists even for a well-specified parametric model.

Basic example:
Markowitz Portfolio choice

Objective: build the best portfolio among a set of n financial assets, for a given known measure of performance, in presence of a risk-free asset.

Recall the classic definition of portfolio optimality (Markowitz 59, 87): *a portfolio is efficient if it has the least risk for a given level of expected return*

Optimization problem:

$$\max_{\theta} \left[\theta'\mu - \frac{\eta}{2}\theta'\Sigma\theta\right] = \max_{\theta} \left[U(\theta)\right]$$

with θ weighting vector, R (n,1)-vector of risky returns with mean μ and variance Σ, η level of risk aversion of the investor.

Solution:

$$\theta^* = \frac{1}{\eta}\Sigma^{-1}(\mu - R_f \iota)$$

$\rightarrow$ This is a 2-fund rule: holding the riskless asset and the tangency portfolio.

Difficulties: optimal rule is **unfeasible** in the sense that the true parameters characterizing the portfolio (here μ and Σ) are unknown.

Naive "plug-in" method (Markowitz 59): true unknown parameters are simply replaced by some estimates.

$\rightarrow$ ER is ignored
$\rightarrow$ Optimality of the estimated rule?

$\Rightarrow$ Can we incorporate ER in the selection method?

A (natural) Bayesian method:

Large literature.
Pioneer work by Zellner and Chetty (1965) and Bawa, Brown and Klein (1979)

Main ideas:

- Choose a prior on the distribution of μ and Σ
- Update this a priori distribution with the data sample and build the *a posteriori* distribution
- Integrate out the parameters over the a posteriori distribution: this gives the predictive distribution

• Maximize the expected utility under the predictive:

$$\begin{aligned}\widehat{\theta}_{Bayes} &= \arg\max_{\theta} \int_{R_{T+1}} \int_{\mu} \int_{\Sigma} U(\theta) p(R_{T+1}, \mu, \Sigma | \Phi_T) d\mu d\Sigma dR_{T+1} \\ &= \arg\max \int_{R_{T+1}} U(\theta) p(R_{T+1} | \Phi_T) dR_{T+1}\end{aligned}$$

where $p(R_{T+1}|\Phi_T)$ is the predictive density

Solution (with a diffuse prior), a 2-fund rule:

$$\widehat{\theta}_{Bayes} = \frac{1}{\eta}\left(\frac{T-N-2}{T+1}\right)\widehat{\Sigma}^{-1}\widehat{\widehat{\mu}}$$

Result: Plug-in estimators methods are outperformed by diffuse prior Bayesian method.

→ Choice of the prior?
→ Closed-form solution?

Non-bayesian methods 1:

General idea: minimize the expected loss of utility resulting in using an approximated rule

$$\theta = \arg\min_{\theta \in R} E_{\widehat{\mu},\widehat{\Sigma}} \left[U(\theta^*) - U(\theta) \right]$$

where θ^* is the unfeasible optimal rule and R the set of all admissible investment rules.

Comments:

i) Consistent from the point of view of the theory of decision

ii) The class of rules is too wide and the problem too hard to be solved

$\rightarrow$ Optimization within a restricted class of rules:

- ter Horst, de Roon and Werker (HRW):
2-fund rules and ER from variance ignored.

- Kan and Zhou (KZ):
a) 2-fund rules with both ER from mean and variance incorporated.
b) 3-fund rules with both ER from mean and variance incorporated.

$\diamond$ **ter Horst, de Roon and Werker (HRW):**

Minimization of a risk function based on the expected utility loss generated by using estimators, for a parametric form of the weighting vector ensuring a 2-fund rule (Σ known):

$$\theta_{HRW} = \arg\min_{\alpha} E_{\widehat{\mu}} \left[U(\theta^*) - U(\widehat{\theta}(\alpha)) \right]$$

$$\text{with } \widehat{\theta}(\alpha) = \frac{1}{\alpha} \Sigma^{-1} \widehat{\widetilde{\mu}}$$

$$\theta_{HRW} = \frac{\theta^2}{\theta^2 + N/T} \frac{1}{\eta} \Sigma^{-1} \widetilde{\mu}$$

$$\text{where } \theta^2 = \widetilde{\mu} \Sigma^{-1} \widetilde{\mu}$$

Note: the optimal rule depends on the unknown parameter (μ), so is unfeasible.
$\rightarrow$ Optimality of the feasible rule?

◇ **Kan and Zhou (KZ):**

Consider a weighting function of the estimated mean and variance where the class of functions of interest is restricted to the 2-fund portfolio rules.
Exploration of a 3-fund rules by investing in addition into the sample global MV portfolio.

$$\theta_{KZ2} = \arg\max_{\kappa_1} E_{\widehat{\mu},\widehat{\Sigma}} \left[U(\frac{\kappa_1}{\eta}\widehat{\Sigma}^{-1}\widehat{\mu}) \right]$$

$$\theta_{KZ2} = \frac{(T-N-4)(T-N-1)}{T(T-2)} \frac{\theta^2}{\theta^2 + N/T} \frac{1}{\eta} \Sigma^{-1}\tilde{\mu}$$

$$\text{where } \theta^2 = \tilde{\mu}'\Sigma^{-1}\tilde{\mu}$$

Result: Bayesian diffuse prior methods are out-performed by unfeasible KZ-2-fund rule

Note: both rules depend on the unknown parameters (μ,Σ), so are unfeasible.
$\rightarrow$ Optimality of the feasible rules?

Non-Bayesian Methods 2:

◇ **Garlappi, Uppal and Wang (GUW):**

Sequential max-min method (Σ known).
1) minimization of the expected utility with respect to the returns falling into a confidence set around the estimated returns
2) maximization of the resulting utility with respect to the weights.

$$\theta_{GUW} = \arg\max_{\theta} \min_{\mu \in CI(\widehat{\mu})} U(\theta)$$

$$\theta_{GUW} = \max\left\{1 - \frac{\sqrt{\epsilon}}{\sqrt{\widehat{\theta}^2}}, 0\right\} \frac{1}{\eta} \Sigma^{-1} \widehat{\widehat{\mu}}$$

with $\epsilon = \dfrac{N\mathcal{F}_{N,T-N}(p)}{T-N}$, $p = .99$, $\widehat{\theta}^2 = \widehat{\widehat{\mu}}' \Sigma^{-1} \widehat{\widehat{\mu}}$

$\longrightarrow$ Is the worst possible scenario optimal from a financial point of view?

Non-Bayesian Methods 3:

◇ **Antoine (A):**

1) Perform a one-sided test that insures that the chosen measure of performance of the portfolio is above a given threshold ie $H_0 : Q_P > c$
2) Maximize the p-value of the above test to get the optimal weights.

Measure of Performance: MV for comparison with literature
Threshold: perf. of the benchmark to beat

Why a test?
1) statistical tool to compare random quantities
2) naturally incorporates ER
3) direct focus on a well-defined objective for a portfolio manager

Restrictions:
- Asset returns are serially independent
- ER coming from the variance is neglected

More formally:

◇ Test: $H_0: \ \mu_P - \eta/2\widehat{\sigma}_P^2 > c$

◇ Test statistic:

$$\mathcal{T} = \frac{\widehat{\mu}_P - \Big(c + \eta/2\widehat{\sigma}_P^2\Big)}{\widehat{\sigma}_P/\sqrt{T}}$$

◇ p-value of the test:

$$p - value = P(\mathcal{T} \leq \frac{\widehat{\mu}_P - \Big(c + \eta/2\widehat{\sigma}_P^2\Big)}{\widehat{\sigma}_P/\sqrt{T}})$$

◇ Maximization Problem:

$$\max_{\theta} \frac{\sqrt{T}[\theta'\widehat{\widetilde{\mu}} - c - \eta/2\theta'\Sigma\theta]}{(\theta'\Sigma\theta)^{1/2}}$$

◇ Solution, 2-fund rule:

$$\theta_A = \frac{\Sigma^{-1}\widehat{\widetilde{\mu}}}{\tilde{\eta}} \quad \text{and} \quad \tilde{\eta} = \eta\frac{\sqrt{U(\widehat{\theta}_{MV})}}{\sqrt{c}}$$

Comparison of investment rules:

1) **Objective function:**
- maximization of some measure of performance of the portfolio (Bayes, HRW, KZ, GUW)
- maximization of the p-value of the test that the performance is above a given threshold

2) **Investment Rule:**

$$\theta = \frac{1}{\tilde{\tilde{\eta}}} \Sigma^{-1} \hat{\tilde{\mu}}$$

where the *corrected* risk-aversion parameter depends on the method used.

a) Feasible 2-fund rules (Bayes, GUW, A) vs Unfeasible rules (KZ, HRW)

b) Natural 2-fund rules (Bayes, GUW, A) vs Imposed 2-fund rules (KZ, HRW)

c) Increase in the risk-aversion parameter $\tilde{\eta} > \eta$ (Bayes, GUW, HRW, KZ) vs $\tilde{\eta}_A >$ or $< \eta$

d) When $T \to \infty$ (ie when ER disappears) the rule tends to the exact MV rule (Bayes, GUW, HRW, KZ) vs no dependence on T (the p-value maximization does not make any sense in absence of ER, since the performance would be deterministic)

3) **Other features:**

a) The p-value framework is consistent with any utility function and not just MV-type utility that has been chosen to compare with the literature.

b) The p-value framework is flexible enough to accommodate for other estimates.

4) **Simulation Exercise**

- 5 risky assets and a riskless asset
- the risky returns follow a multivariate normal distribution
- the true model parameters obtained from monthly unhedged returns of stock indices for the G5 countries over the period January 1974 to December 1998.

	Moyenne	Std
France	0.014	0.069
Germany	0.013	0.059
Japan	0.011	0.067
UK	0.015	0.073
USA	0.012	0.044

$$\rho_0 = \begin{pmatrix} 1 & .590 & .390 & .541 & .456 \\ & 1 & .338 & .424 & .347 \\ & & 1 & .342 & .221 \\ & & & 1 & .506 \\ & & & & 1 \end{pmatrix}$$

Estimators calculated with a rolling window of size T.

We compare:
- A for 3 different choices of the threshold A1, A2 and A3 with resp. $c = 1.1R_f,\ 1.2R_f,\ 1.3R_f$;
- GUW;
- KZ: feasible 2-fund rule (KZ2) and feasible 3-fund rule (KZ3)

Criteria of Comparison:

i) Stability of the positions trough time via transaction costs
ii) Profitability of the method via average utility

i) Stability of the rules via the transaction costs:

T	A1	A2	A3	KZ	KZ3	GUW	MV
60	**14.16**	20.02	24.52	33.35	33.96	**3.82**	69.52
120	**8.15**	11.52	14.11	20.44	17.29	**4.90**	31.64
180	**5.94**	8.41	10.29	15.01	12.06	**5.09**	20.58
240	**4.65**	6.57	8.05	11.93	9.19	**4.85**	15.18
300	**3.73**	5.28	6.46	9.98	7.66	**4.64**	12.06

ii) Mean utility over the whole period:

T	A1	A2	A3
60	**0.0009**	0.0009	0.0008
120	0.0013	0.0015	**0.0016**
180	0.0015	0.0018	**0.0019**
240	0.0016	0.0020	**0.0022**
300	0.0017	0.0021	**0.0023**

KZ	KZ3	GUW	MV
-0.0003	-0.0003	0.0001	-0.0085
0.0009	0.0010	0.0006	-0.0015
0.0014	0.0015	0.0010	0.0003
0.0019	0.0017	0.0014	0.0012
0.0021	0.0018	0.0017	0.0016

To go further:

Parameter uncertainty implicitly supposed than the reference model is correct - ER is really the risk coming from the estimation of the parameter of the (true) model.

$\rightarrow$ what if there is uncertainty about the model?

Some work has been done to develop robust methods to model risk (MR)

- Local deviation (*à la Huber*): you consider some reference model; then you incorporate some perturbation; typically you some probability distribution on several models.
See Cavadini, Sbuelz, Trojani: extension of HRW with incorporation of MR via local deviation

- GUW: direct extension of their methods to incorporate MR when the investor relies on a factor model to estimate the returns
1) minimization of the expected utility with respect to the means of the assets and the factors falling into a confidence set around the estimated ones
2) maximization of the resulting utility with respect to the weights.

- Model averaging: use several models to deduce the associated optimal rule and then average over the rules.

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