



CENTRE NATIONAL
DE LA RECHERCHE
SCIENTIFIQUE



FINANC ONCEPTS

Hedging options in presence of jumps

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- Hedging strategies and residual risk.
- Pathwise decomposition of P&L
- Hedging with the underlying:
 - Δ -neutral hedging and its shortcomings
 - Hedging with Black Scholes Δ
 - Quadratic hedging: minimizing variance of P&L
- Hedging with options:
 - Δ/Γ -neutral hedging
 - Hedging with Black Scholes Δ and Γ
 - Quadratic hedging with options
 - Some limiting cases
- Comparison of risk profiles of different hedging strategies

Discontinuities in asset price dynamics

Ample econometric evidence for presence of jumps in the price process:

Evidence from time series of underlying:

Barndorff Nielsen & Shepard (2004,2005), Chernov et al (2003), Cont & Mancini (2005), Eraker, Johannes, Poulson (2003), Bollerslev et al. (2005)

Evidence from option prices (short term options):

Andersen & Andreasen (2001), Bates (1999), Carr & Wu (2003), Cont & Tankov (2004)

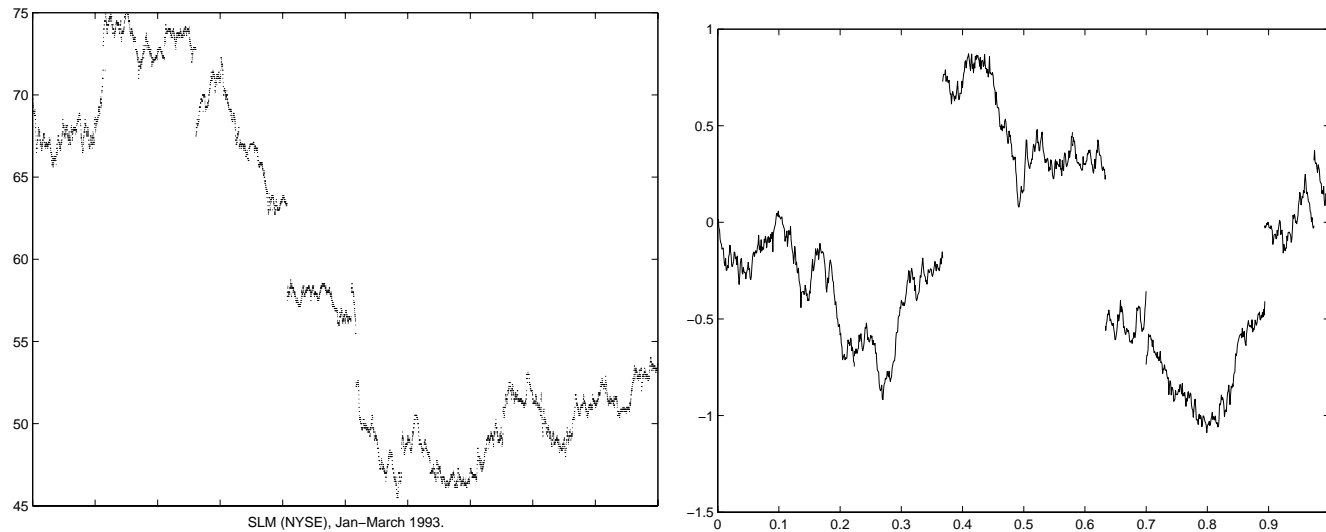


Figure 1: Jumps account for a large proportion of returns over short horizons. Left: SLM (NYSE), Jan-Mars 1993. Right: path of a Lévy process.

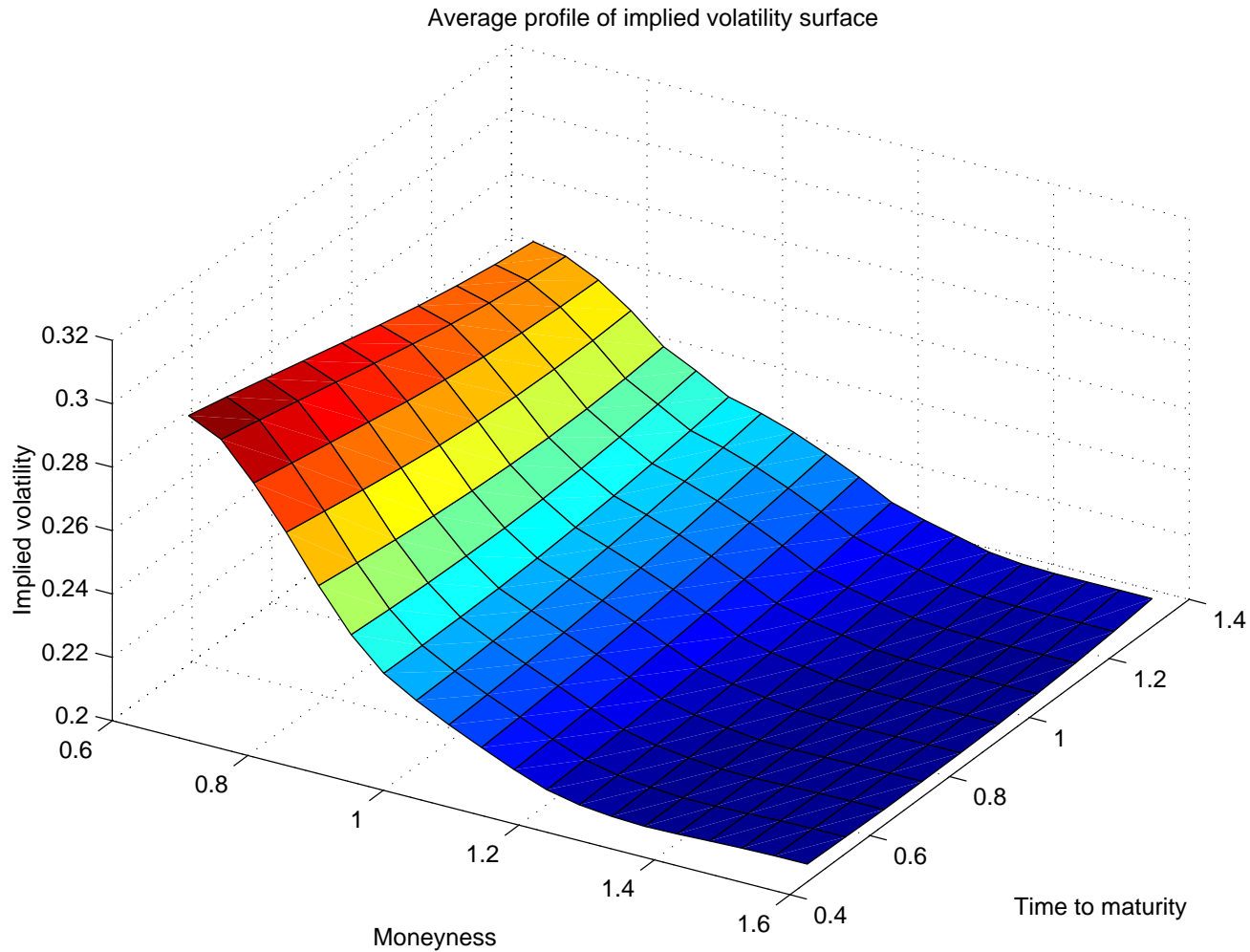


Figure 2: Jumps are important for understanding prices of short term options. Implied volatilities of DAX index options, 2001.

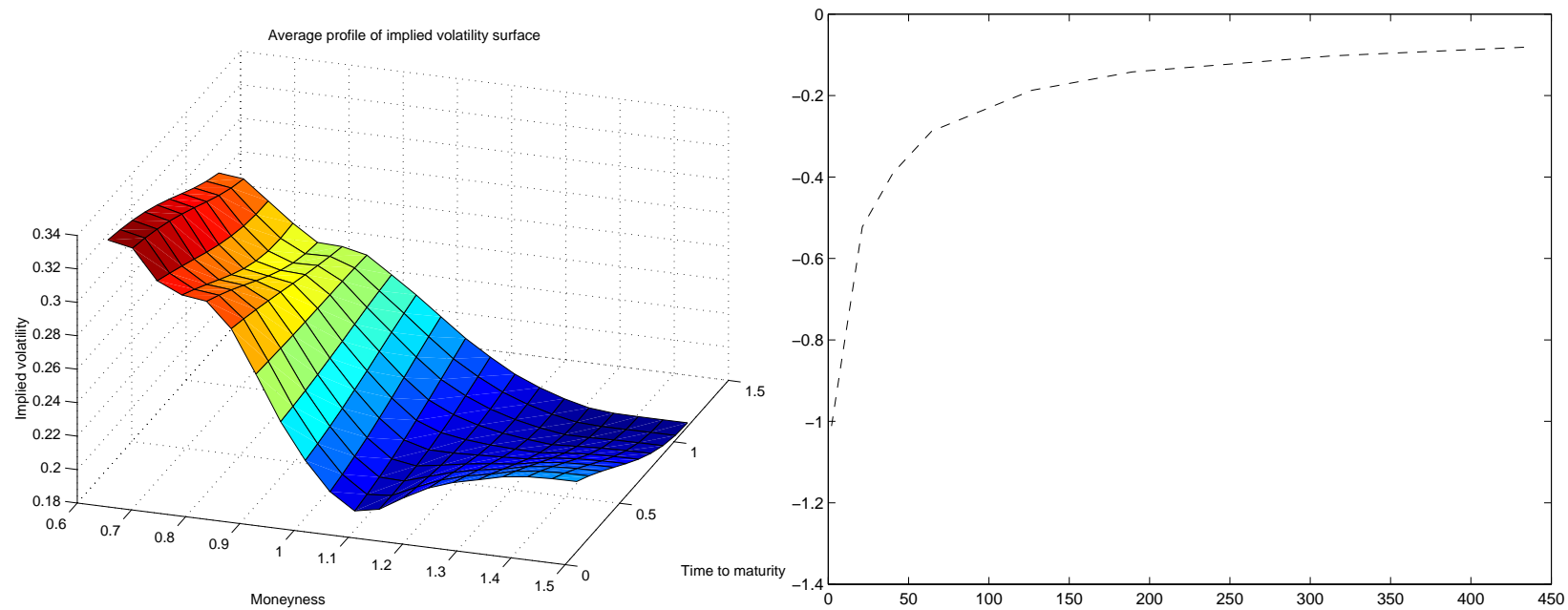


Figure 3: Left: The implied volatility surface for S&P 500 options. Right: At-the-money skew $\frac{\partial \Sigma}{\partial K}(K = S, T)$ as a function of the maturity T (in days).

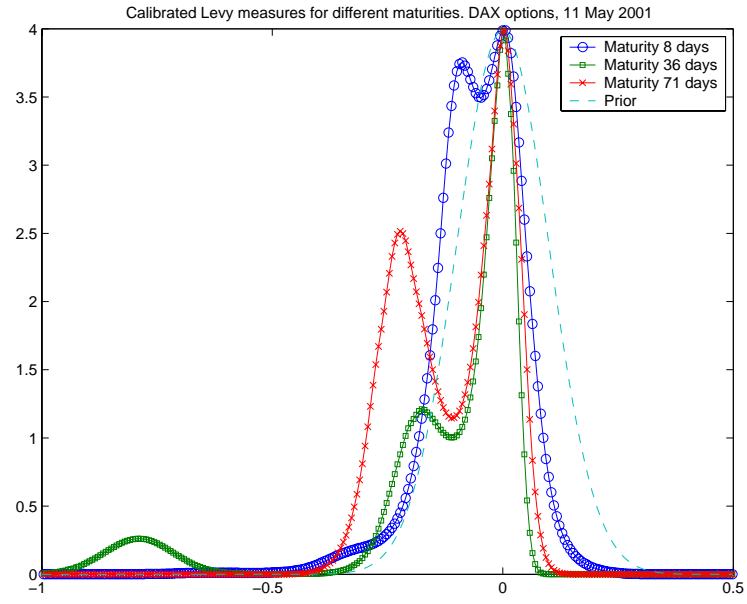
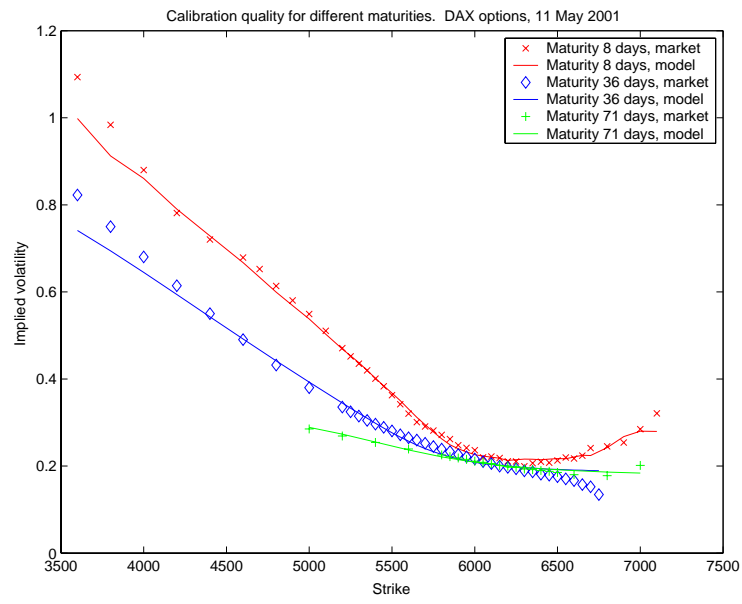


Figure 4: Lévy measures implied from prices of DAX options, May 2001 (Cont & Tankov, 2004).

“Price discontinuity and concentration are major ingredients in a realistic evaluation of risks.”

B.B. Mandelbrot *Fractals and Scaling in Finance: Discontinuity, Concentration, Risk.*, 1997.

Many **pricing** models with jumps have been proposed in the literature: Black Scholes+ compound Poisson jumps, exponential Lévy models, SV models with jumps, ... but little discussion about **hedging**. What are the consequences of such price discontinuities for hedging?

Hedging in diffusion models

$$\begin{aligned} dS_t &= \mu_t S_t dt + \sigma(t, S_t) dW_t \\ H_T &: \text{option payoff} \quad H_t = C(t, S_t) : \text{option value} \end{aligned} \quad (1)$$

Main intuitions conveyed by diffusion models:

- Hedging = replication: if $E[H_T^2] < \infty$ there exists a perfect hedging strategy ϕ_t such

$$H_T = c + \int_0^T \phi_t^0 dS_t \quad (2)$$

- The return on the hedged position does not depend on μ_t .
- Hedge ratio = sensitivity to underlying: $\phi_t^0 = \frac{\partial C}{\partial S}(t, S_t)$
- Options are redundant with respect to the underlying; though they may be used as hedging instruments, their use does not allow in principle to improve hedging performance.

These ideas generalize to diffusion-type stochastic volatility models

$$dS_t = \mu_t S_t dt + S_t \sigma_t dW_t^1 \quad (3)$$

$$\sigma_t = f(Y_t) \quad dY_t = b(t, S_t, Y_t) dt + \gamma(t, Y_t) dW_t^2 \quad (4)$$

by including **one** liquid call/put option C_t^1 as a hedging instrument:

- There exists a perfect hedging strategy (ϕ^0, ϕ^1) such that

$$H_T = c + \int_0^T \phi_t^0 dS_t + \int_0^T \phi_t^1 dC_t^1 \quad (5)$$

- The return on the hedged position does not depend on μ_t .
- Hedge ratios can be obtained from sensitivities to S, Y :

$$\phi_t^1 = -\frac{\frac{\partial C}{\partial y}(t, S_t, Y_t)}{\frac{\partial C^1}{\partial y}(t, S_t, Y_t)} \quad \phi_t^0 = \frac{\partial C}{\partial S}(t, S_t, Y_t) - \phi_t^1 \frac{\partial C^1}{\partial S}(t, S_t, Y_t)$$

- In principle, any option can be used as hedging instrument.

- According to the model, using multiple options does not allow to improve hedging performance.

Questions

- How should hedging strategies be modified to take into account jumps in prices?
- Which of the intuitions conveyed by diffusion models are robust to the presence of jumps?

Hedging in presence of jumps

In presence of jumps, perfect replication of a derivative may not be possible by trading in the underlying (“delta hedging”).

Hedging has to be understood, in more realistic terms, as an *approximation* problem: approximating a target payoff using a portfolio of underlying asset + liquid options.

Standard options are non-redundant so their use for hedging is natural.

Various approximation criteria will then lead to different hedging strategies and different risk profiles for the hedged position.

Each approximation leads to a hedging error: a random P&L which has some *residual risk*.

Typically: minimize some measure of terminal shortfall

$$\inf_{\phi} E^{\mathbb{P}}[l(c + \int_0^T \phi_t^0 \cdot dS_t + \int_0^T \phi_t \cdot dC_t^* - H_T)] \quad (6)$$

H_T : option payoff

$C_t^* = (C_t^i, i = 1..n)$: market prices of liquid options used as hedging instruments

ϕ_t^0 : position in underlying

$\phi_t = (\phi_t^1, \dots, \phi_t^n)$: position in options

$V_T = c + \int_0^T \phi_t \cdot dS_t + \int_0^T \phi_t \cdot dC_t^*$: value of hedging portfolio.

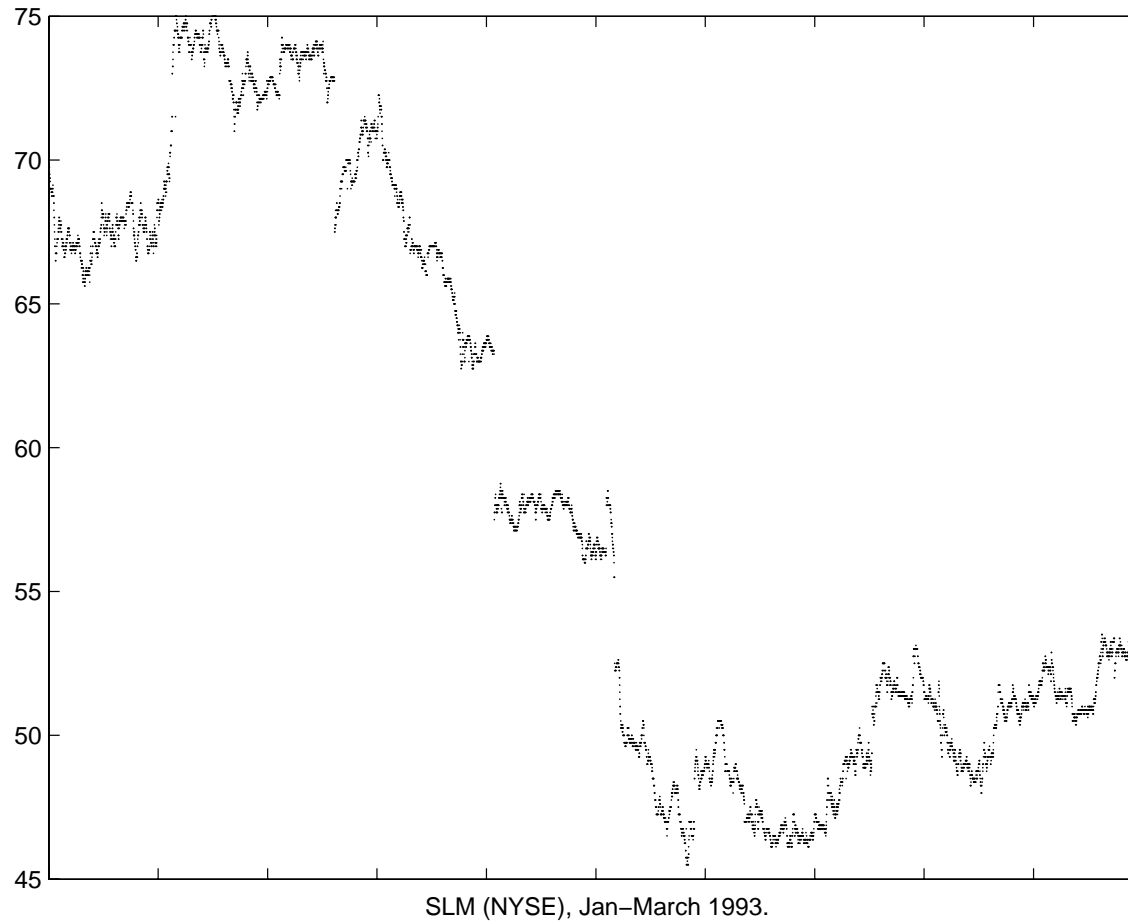
Choice of loss function:

- Quadratic $l(x) = x^2$: tractable, leads to a linear pricing/hedging (consistent with market practice for standard options).
- Exponential $l(x) = \exp(-\gamma x)$: gain/loss asymmetry, link with entropy but leads to **nonlinear pricing and hedging**.
- Other utility functions: loss of tractability (nonlinear PDEs), hedges depends on initial wealth, nonlinear hedging.

Choice of probability measure:

- \mathbb{P} = 'objective' probability measure: makes sense but may be difficult to identify, hedging strategy will depend on anticipated returns (directional position).
- \mathbb{Q} = risk neutral probability: can be identified by "calibrating" the model to observed market prices of options, hedging strategy will only depend on anticipations on "volatility".

Description of jumps in prices



Need to describe **position** and **amplitudes** of jumps

Description of jumps of a discontinuous process

Model: $(X_t)_{t \in [0, T]}$ right continuous with left limits (cadlag)

$X_t = \lim_{s \downarrow t} X_s$ but one can have “unexpected” jumps

$$X_t \neq X_{t-} = \lim_{s \uparrow t} X_s$$

Jump size $\Delta X_t := X_t - X_{t-}$

Jump measure of X :

$$J_X = \sum_{s \in [0, t], \Delta X_s \neq 0} \delta_{s, \Delta X_s} \quad (7)$$

$$\forall A \subset \mathbb{R}^d - \{0\}, \quad J_X([0, t] \times A) = \#\{s \in [0, t], \Delta X_s \in A\}$$

Integration with respect to J_X :

$$\int_0^t \int_{\mathbb{R}} f(s, x) J_X(ds dx) = \sum_{s \in [0, t], \Delta X_s \neq 0} f(s, \Delta X_s)$$

The positions and sizes of jumps is often modeled by a Poisson random measure J_X

$\forall A \subset \mathbb{R}^d - \{0\}$, $N_t(A) = J_X([0, t] \times A)$ is a Poisson process with intensity $\nu(A)$ where ν is the Lévy measure of X .

$$\nu(A) = \frac{1}{T} E[\#\{s \in [0, T] \Delta X_s \in A\}]$$

Compensated Poisson measure:

$$\tilde{J}_X([0, t] \times A) = J_X([0, t] \times A) - E^{\mathbb{P}}(J_X([0, t] \times A)) \quad (8)$$

$M_t = \int_0^t \int_{\mathbb{R}} f(s, x) \tilde{J}_X(ds dx)$ is the "martingale part" (unpredictable component) of $\sum_{s \in [0, t]} f(s, \Delta X_s)$

Ito formula for a discontinuous (cadlag) process

For any $C^{1,2}$ function $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$,

$$\begin{aligned}
 f(t, X_t) - f(0, X_0) &= \sum_{\substack{0 \leq s \leq t \\ \Delta X_s \neq 0}} [f(s, X_s) - f(s, X_{s-}) - \Delta X_s \frac{\partial f}{\partial x}(s, X_{s-})] \\
 &+ \int_0^t \frac{\partial f}{\partial s}(s, X_s) ds + \int_0^t \frac{\partial f}{\partial x}(s, X_{s-}) dX_s + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_{s-}) d[X, X]_s^c \\
 &= \int_0^t \frac{\partial f}{\partial s}(s, X_s) ds + \int_0^t \frac{\partial f}{\partial x}(s, X_{s-}) dX_s + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_{s-}) d[X, X]_s^c \\
 &\quad + \int_0^t \int_{\mathbb{R}} J_X(ds dz) [f(s, X_{s-} + z) - f(s, X_{s-}) - z \frac{\partial f}{\partial x}(s, X_{s-})]
 \end{aligned}$$

If the jumps have finite variation

$$\sum_{0 \leq t \leq T} |\Delta X_t| < \infty$$

then jumps need not be compensated:

$$X_t = X_t^c + \sum_{s \leq t} \Delta X_s$$

and we obtain a simpler formula

$$\begin{aligned} f(t, X_t) - f(0, X_0) &= \sum_{\substack{0 \leq s \leq t \\ \Delta X_s \neq 0}} [f(s, X_s) - f(s, X_{s-})] \\ &+ \int_0^t \frac{\partial f}{\partial s}(s, X_s) ds + \int_0^t \frac{\partial f}{\partial x}(s, X_{s-}) dX_s^c + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_{s-}) d[X, X]^c_s \end{aligned}$$

Dynamics of the underlying asset

$$\frac{dS_t}{S_{t-}} = \sigma_{t-} dW_t + \int (e^z - 1) \tilde{J}_X(dt dz) \quad (9)$$

$$S_t = S_0 + \int_0^t S_{u-} \sigma(u, S_{u-}) dW_u + \int_0^t \int_{\mathbb{R}} \tilde{J}_X(du dz) S_{u-} (e^z - 1)$$

$\tilde{J}_X(dt dz) = J_X(dt dz) - dt \nu_t(dz)$: \mathbb{Q} -compensated jump measure.

Allows for local volatility, stochastic volatility and/or time-dependence in the Lévy density (intensity and distribution of jumps).

Includes: local vol+ jumps (Andersen Andreasen), jumps in volatility (Bates, BNS), time inhomogeneity.

Quadratic variation:

$$[S]_t = \underbrace{\int_0^t S_u^2 \sigma_u^2 du}_{[S]_t^c} + \sum_{0 \leq u \leq t} |\Delta S_u|^2 \quad (10)$$

Markovian (one-factor) case: local vol + jumps

$$\frac{dS_t}{S_{t-}} = \sigma(t, S_{t-})dW_t + \int (e^z - 1)\tilde{J}_X(dtdz) \quad (11)$$

$$S_t = S_0 + \int_0^t S_{u-}\sigma(u, S_{u-})dW_u + \int_0^t \int_{\mathbb{R}} \tilde{J}_X(dudz) S_{u-}(e^z - 1)$$

The value of a terminal payoff $H(S_T)$ is given by $C(t, S)$ where:

$$C : [0, T] \times [0, \infty) \rightarrow \mathbb{R} \quad (12)$$

is a solution of the partial integro-differential equation:

$$\begin{aligned} & \frac{\partial C}{\partial t}(t, S) + rS \frac{\partial C}{\partial S}(t, S) + \frac{\sigma(t, S)^2 S^2}{2} \frac{\partial^2 C}{\partial S^2}(t, S) - rC(t, S) \\ & + \int \nu_t(dx) [C(t, Se^x) - C(t, S) - S(e^x - 1) \frac{\partial C}{\partial S}(t, S)] = 0 \end{aligned} \quad (13)$$

on $[0, T) \times (0, \infty)$ with the terminal condition:

$$\forall S > 0, \quad C(T, S) = H(S). \quad (14)$$

Exponential Lévy models

$S_t = S_0 \exp(rt + X_t)$ where $X_t =$ Lévy process: $\sigma(t, S) = \sigma, \nu_t = \nu$

Finite activity: $\lambda = \nu(\mathbb{R}) < \infty$

- Poisson jumps: $\nu(dx) = \sum_{k=1}^n p_k \delta_{y_k}(dx)$
- **Merton model:** $\nu(x) = \frac{\lambda}{\sqrt{2\pi\gamma^2}} e^{-(x-\mu)^2/2\gamma^2}$
- **Kou :** $\nu(x) = p\alpha_1 e^{-\alpha_1 x} 1_{x>0} + (1-p)\alpha_2 e^{+\alpha_2 x} 1_{x<0}$

Infinite activity:

- **Variance Gamma** (Madan & Milne):
 $\nu(x) = A|x|^{-1} \exp(-\eta_{\pm}|x|)$
- Tempered stable process: $\nu(x) = A_{\pm}|x|^{-(1+\alpha)} \exp(-\eta_{\pm}|x|)$
- (Generalized) hyperbolic processes (Eberlein et al.)

Hedging strategies and residual risk

Consider $k + 1$ options on S_t with payoffs $H_T^i, i = 0..k$ and value $C_t^i = C^i(t, S_t)$. Operator sells an option with payoff $H_T^0 = H_T$

Hedging instruments: underlying + k liquid options

$$dV_t = \phi_t^0 dS_t + \sum_{i=1}^k \phi_t^i dC_t^i \quad (15)$$

Hedging error/ P&L:

$$\epsilon(\phi)_T = C_0 + \int_0^T \phi_t^0 dS_t + \int_0^T \sum_{i=1}^k \phi_t^i dC_t^i - H_T \quad (16)$$

$$dC_t^i = \frac{\partial C_i}{\partial S}(t, S_{t-}) dS_t + \frac{\partial C_i}{\partial t} dt + \frac{\sigma^2(t, S_t) S_t^2}{2} \frac{\partial^2 C_i}{\partial S^2} dt \quad (17)$$
$$+ [C_i(t, S_t) - C_i(t, S_{t-}) - \frac{\partial C_i}{\partial S}(t, S_{t-}) \Delta S_t]$$

Pathwise decomposition of profit and loss (P&L)

The hedging error is a path-dependent random variable.

Ito formula \Rightarrow decomposition of hedging error along the path:

$$\begin{aligned} \epsilon(\phi)_T = & - \int_0^T \left(\frac{\partial C(t, S_{t-})}{\partial S} - \phi_t^0 - \sum_{i=1}^M \phi_t^i \frac{\partial C^i(t, S_{t-})}{\partial S} \right) \sigma(t, S_{t-}) S_{t-} dW_t \\ & - \int_0^T \int_{\mathbb{R}} (C(t, e^z S_{t-}) - C(t, S_{t-}) - \\ & \sum_{i=1}^M \phi_t^i (C^i(t, e^z S_{t-}) - C^i(t, S_{t-})) - (e^z - 1) S_{t-} \phi_t) \tilde{J}_X(dt dz) \quad (18) \end{aligned}$$

Residual risk

Risk-neutral variance $E^{\mathbb{Q}}[(\epsilon(\phi)_T)^2]$ can be computed using the isometry formula:

$$\begin{aligned}
 E^{\mathbb{Q}}[(\epsilon(\phi)_T)^2] &= \int_0^T E^{\mathbb{Q}} \left[\left(\frac{\partial C(t, S_{t-})}{\partial S} - \phi_t \right)^2 S_t^2 \right] \sigma^2 dt \\
 &+ \int_0^T \int_{\mathbb{R}} E^{\mathbb{Q}} \left[(C(t, e^z S_t) - C(t, S_t) - (e^z - 1)S_t \phi_t)^2 \right] dt \nu(dz),
 \end{aligned} \tag{19}$$

Under the historical measure \mathbb{P} , P&L may have non-zero expectation:

$$\begin{aligned}
 E^{\mathbb{P}}[\epsilon(\phi)_T] &= \int_0^T \left(\frac{\partial C(t, S_t)}{\partial S} - \phi_t \right) (\mu - r) S_{t-} dt \\
 &+ \int_0^T \int_{\mathbb{R}} dt (\nu_t^{\mathbb{P}} - \nu_t^{\mathbb{Q}})(dz) E^{\mathbb{P}} \left[(C(t, e^z S_{t-}) - C(t, S_{t-}) - (e^z - 1)S_{t-} \phi_t) \right]
 \end{aligned}$$

Comparing hedging strategies

- Hedging with the underlying:
model Δ -neutral hedge,
quadratic hedging,
Black-Scholes Δ with implied vol.
- Static hedging with options.
- Dynamics hedging with underlying+ options:
model Δ/Γ -neutral hedge
variance minimization by dynamics trading in underlying+
options
Black-Scholes Δ/Γ -neutral with implied vol.

Each strategy ϕ leads to certain P&L $\epsilon(\phi)$:

What does the distribution of the P&L look like?

How large is the residual risk $\text{var}(\epsilon(\phi))$?

Hedging with the underlying

$$\begin{aligned} \epsilon(\phi)_T = & - \int_0^T \left(\frac{\partial C(t, S_{t-})}{\partial S} - \phi_t^0 \right) \sigma(t, S_{t-}) S_{t-} dW_t \\ & - \int_0^T \int_{\mathbb{R}} (C(t, e^z S_{t-}) - C(t, S_{t-}) - (e^z - 1) S_{t-} \phi_t) \tilde{J}_X(dt dz) \end{aligned} \quad (20)$$

Choices for ϕ_t^0 :

Model Δ : $\phi_t^0 = \frac{\partial C}{\partial S}(t, S_{t-})$

Black Scholes Δ : $\phi_t^0 = \frac{\partial C^{BS}}{\partial S}(t, S_{t-}; \Sigma_t)$ where $\Sigma_t =$ implied vol

Quadratic hedge: $\phi^0 = \arg \min_{\phi} E[|\epsilon(\phi)_T|^2]$

Black Scholes Δ -hedge

In case the option being hedged has a market price, one can also use a Black-Scholes Δ based on implied volatility

Let $C^{BS}(t, S; \sigma)$ be the Black Scholes price for the option.

- Observe price of option C_t in the market.
- Back out implied volatility Σ_t from market price:

$$C_t = C^{BS}(t, S; \Sigma_t) \quad (21)$$

- Compute a Black-Scholes delta using Σ_t

$$\phi_t = \frac{\partial C^{BS}}{\partial S}(t, S; \Sigma_t) \quad (22)$$

Widely used in practice by traders.

Delta-hedging with implied volatility: case of exponential Lévy models

Consider the exponential-Lévy model $S_t = S_0 \exp X_t$ where X Lévy process under \mathbb{Q} . Then:

- Implied volatility of call options is given by $\Sigma_t(T, K, t, S_t) = I(\frac{K}{S_t}, T - t)$ where $I : (m, \tau) \rightarrow I(m, \tau)$ is the initial implied volatility surface parameterized in terms of moneyness $m = K/S_t$ and time to maturity $\tau = T - t$.
- Implied volatility will jump if and only if the underlying jumps and the size of jumps in implied volatility will be a function $j(t, S_{t-}, \Delta S_t)$ of the jumps in the underlying, defined by:

$$C(t, S + \Delta S) - C(t, S) = C^{BS}(t, S + \Delta S, \Sigma + j(t, S, \Delta S)) - C^{BS}(t, S, \Sigma)$$

Proposition 1 (Black-Scholes Δ -hedge in an exponential Lévy model)

Consider an exponential Lévy model $S_t = S_0 \exp(X_t)$ where the initial implied volatility surface is given by

$$\Sigma_0(K, T) = I\left(\frac{K}{S_0}, T\right) \quad (23)$$

Assume the value function of the option is a smooth solution of the pricing PIDE. Then the residual risk of the Black Scholes delta-hedging strategy is given by:

$$\begin{aligned} \epsilon(\phi^{\text{BS}}) = V_T - H_T = & \int_0^T \frac{K}{S_{t-}^2} \frac{\partial C^{\text{BS}}}{\partial \sigma}(t, S_{t-}; \Sigma_t) \frac{\partial I}{\partial m}\left(\frac{K}{S_{t-}}, T - t\right) dS_t - \quad (24) \\ & \int \tilde{J}_X(dt dz) \{C^{\text{BS}}(t, S_{t-}e^z, \Sigma_{t-} + j(t, S_{t-}, S_{t-}(e^z - 1))) - C^{\text{BS}}(t, S_{t-}, \Sigma_{t-}) \\ & - S_{t-}(e^z - 1)[\Delta^{\text{BS}}(t, S_{t-}; \Sigma_{t-}) - \frac{K}{S_{t-}^2} \frac{\partial C^{\text{BS}}}{\partial \sigma}(t, S_{t-}; \Sigma_t) \frac{\partial I}{\partial m}\left(\frac{K}{S_{t-}}, T - t\right)]\} \end{aligned}$$

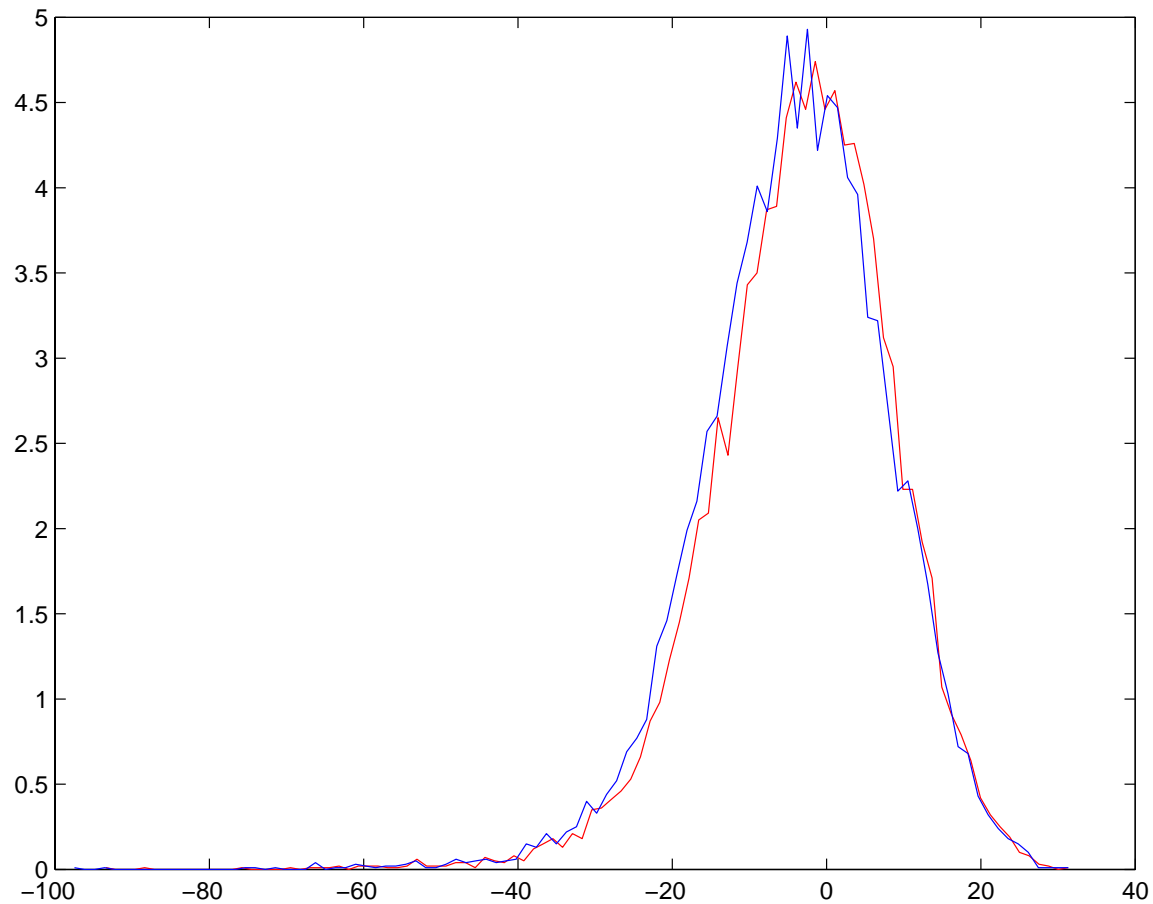


Figure 5: Histogram of hedging errors of a Black Scholes delta hedging strategy for a call option, as percentage of initial selling price, in a Merton model.

Delta-hedging with implied volatility: remarks

- In case of a downward sloping skew, the first term in (24) has sign opposite to the move in the stock price:

$$\frac{K}{S_{t-}^2} \frac{\partial C^{BS}}{\partial \sigma}(t, S_{t-}; \Sigma_t) \frac{\partial I}{\partial m} \left(\frac{K}{S_{t-}}, T - t \right) < 0 \quad (25)$$

so implied volatility delta “under-hedges” the option i.e. leaves a residual exposure to the underlying *even in absence of jumps*.

- In the event of a jump in S_t , the impact on the P&L is

$$C^{BS}(t, S_{t-} + \Delta S_t, \Sigma_{t-} + j(t, S_{t-}, \Delta S_{t-})) - C^{BS}(t, S_{t-}, \Sigma_{t-}) - \Delta S_{t-} \Delta^{BS}(t, S_{t-}; \Sigma_{t-}) \simeq \frac{\Delta S_{t-}^2}{2} \Gamma^{BS} + j(t, S_{t-}, \Delta S_{t-}) \frac{\partial C^{BS}}{\partial \sigma} \quad (26)$$

so residual risk has two main contributions: a “Vega” risk which results from the jump in implied volatility, and a convexity (Gamma). This suggests using options for further hedging these components.

Δ hedging

Idea: choose $\phi_t = \frac{\partial C}{\partial S}(t, S_{t-})$ (Merton 1976)

Hedging loss:

$$\epsilon(\phi) = \int_0^T \int_{\mathbb{R}} \left(C(t, e^z S_{t-}) - C(t, S_{t-}) - (e^z - 1) S_{t-} \frac{\partial C}{\partial S}(t, S_{t-}) \right) \tilde{J}_X(dt dz) \quad (27)$$

Residual risk = martingale (unpredictable) part of P&L due to jumps.

Variation of the option due to the diffusion is completely hedged but jump risk is left totally unhedged. Merton (1976): “jump risk can be diversified across assets”.

Hedging bias:

Risk neutral intensity and distribution of jumps can be different from historical one: risk-neutral distribution of jumps is typically more negatively skewed.

This results in a hedging bias: $\epsilon(\phi) =$

$$\int_0^T dt \int_{\mathbb{R}} [\nu^{\mathbb{Q}}(dz) - \nu^{\mathbb{P}}(dz)] \underbrace{\left(C(t, e^z S_{t-}) - C(t, S_{t-}) - (e^z - 1) S_{t-} \frac{\partial C}{\partial S}(t, S_{t-}) \right)}_{>0 \text{ for convex payoffs}} dt$$

→ systematic bias in hedging error for convex payoffs

A bound on residual risk

Proposition 2 *The residual for the Δ -hedged position can be estimated as follows:*

$$\begin{aligned} \int_{\mathbb{R}} E^Q \left[\left(C(t, e^z S_t) - C(t, S_t) - (e^z - 1) S_t \frac{\partial C(t, S_t)}{\partial S} \right)^2 \right] \nu^Q(dz) \\ \leq \frac{K^2 \int_{-\infty}^{\infty} (e^z - 1)^4 \nu^Q(dz)}{8\sigma^2 \pi(T - t)} \end{aligned} \quad (28)$$

For short maturities a better bound is:

$$\begin{aligned} \int_{-\infty}^{\infty} E^Q \left[\left(C(t, e^z S_t) - C(t, S_t) - (e^z - 1) S_t \frac{\partial C(t, S_t)}{\partial S} \right)^2 \right] \nu^Q(dz) \\ \leq E^Q[S_t^2] \int_{-\infty}^{\infty} (e^x - 1)^2 \nu^Q(dz) = S_0^2 e^{t(\sigma^2 + \nu^Q((e^x - 1)^2))} \nu^Q((e^x - 1)^2) \end{aligned} \quad (29)$$

Δ - hedging: advantages and drawbacks

Advantage: simple to understand, analogy with Black Scholes.

Drawbacks:

- Jump risk unhedged. Merton's argument (diversifiability of jump risk) fails in equity markets.
- For short term options, Δ -hedging may actually **increase** risk!
- Systematic bias in P&L due to unhedged gamma risk.
- In pure jump models like Variance Gamma, options may not have smooth dependence in underlying: Δ may be infinite! (Cont & Voltchkova, 2005)
- What are we actually minimizing?

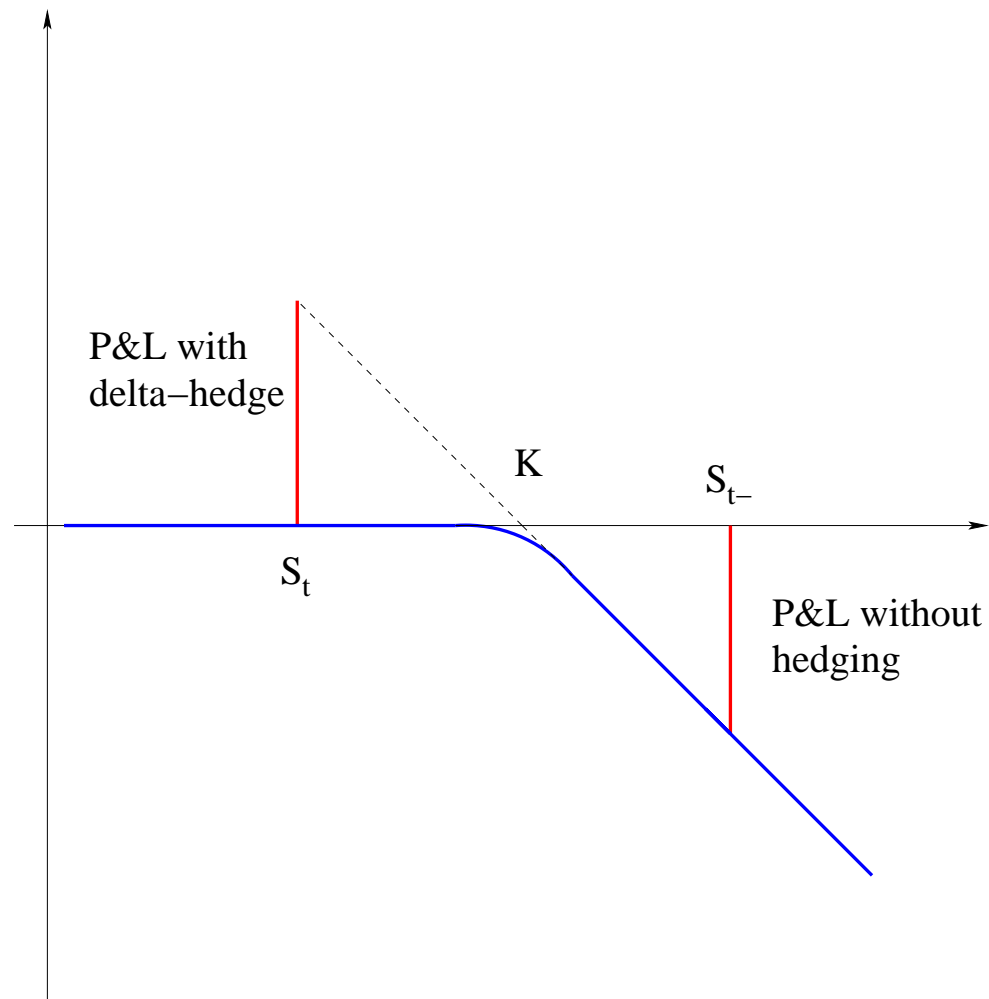


Figure 6: Effect of delta-hedging on P&L in presence of jumps close to maturity. Delta-hedging need not reduce jump risk compared to no hedging at all.

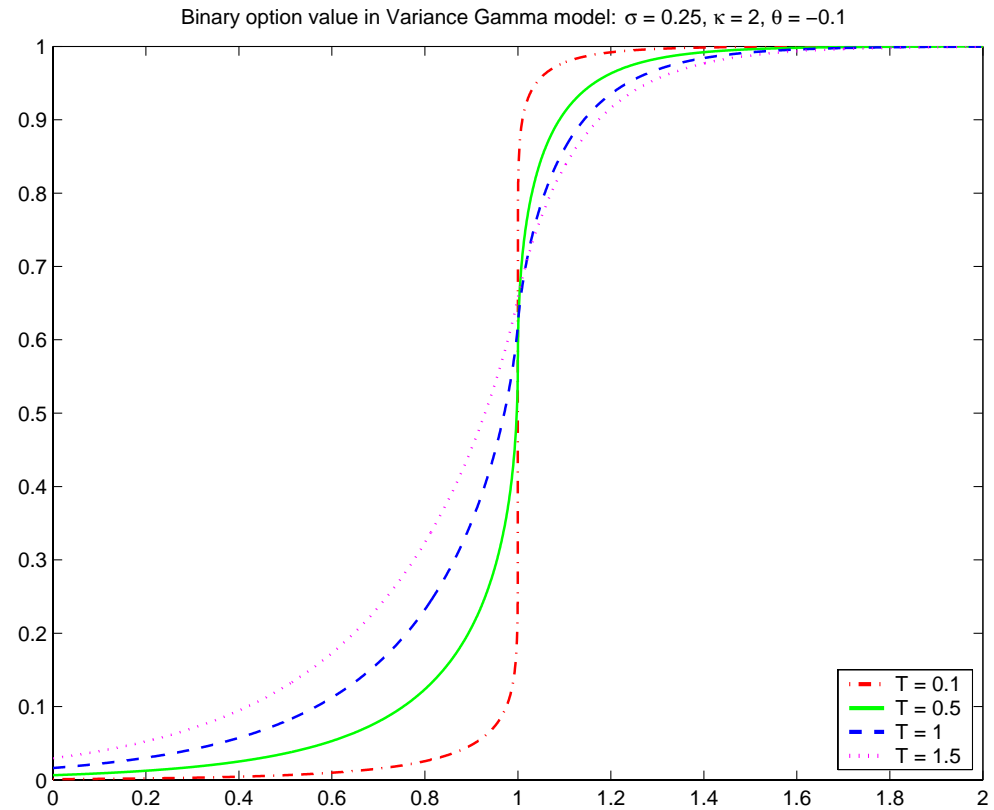


Figure 7: In pure jump models like Variance Gamma, options may not have smooth dependence in underlying: Δ may not exist!

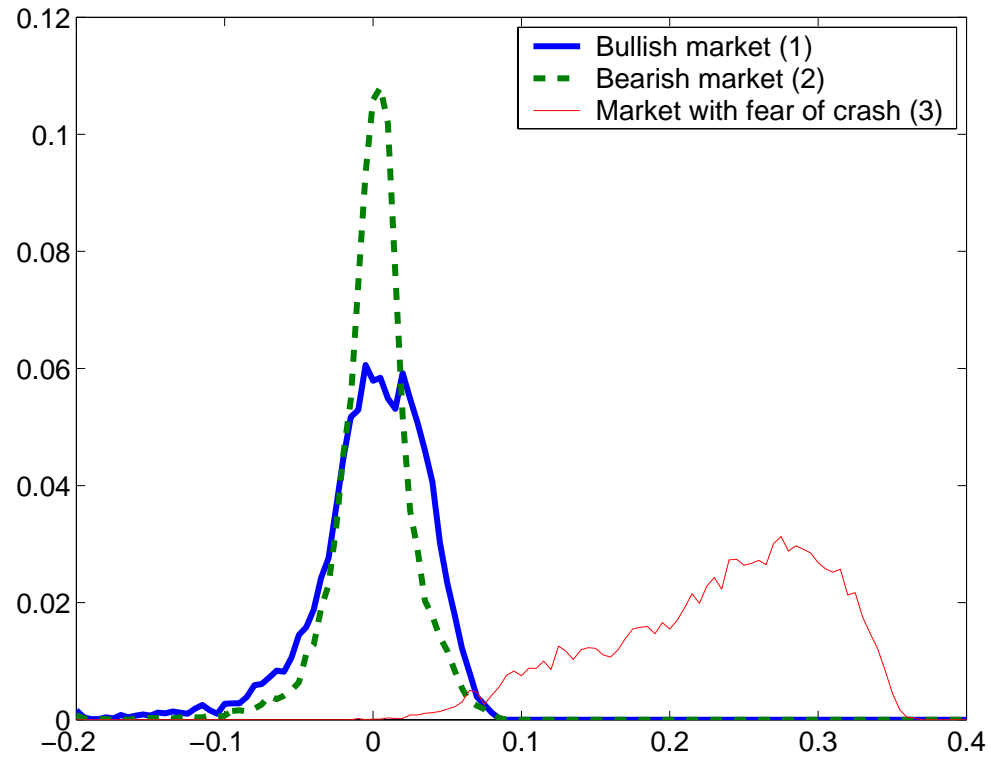


Figure 8: Performance of the delta hedging strategy in different markets with jumps: P&L distributions.

	μ	σ	λ	Jump mean	Jump stdev
Model 1:	—	0.1	5	−0.05	0.1
Bullish market	0.2	0.1	5	−0.05	0.1
Model 2:	—	0.1	5	−0.05	0.1
Bearish market	−0.2	0.1	5	−0.05	0.1
Model 3:	—	0.1	10	−0.2	0.2
Fear of crash	0.2	0.1	5	−0.05	0.1

Table 1: Parameters of models used for numerical experiments.

	Model 1: bullish market	Model 2: bearish market	Model 3: fear of crash
$\overline{P\&L}$	8.0×10^{-5}	3.4×10^{-4}	0.233
$\sqrt{\overline{P\&L^2}}$	0.0387	0.0274	0.2439
$\sqrt{E[P\&L^2]} \leq$	0.1832	0.1641	—
Unhedged $\sqrt{\overline{P\&L^2}}$	0.2209	0.2650	0.3736

Table 2: Performance of delta-hedging strategies in different markets with jumps: average P&L, theoretical bounds and comparison to unhedged portfolios.

Hedging jumps of a given size

Idea: hedge against a typical crash of given size $\Delta S_t = S_{t-} x$.

Choose

$$\phi_t^0 = \frac{C(t, S_{t-}(1+x)) - C(t, S_{t-})}{xS_{t-}}$$

Hedge ratio= finite difference, but not necessarily with small x

This choice hedges completely the loss given a jump of $x\%$ but leaves the position unhedged with respect to jumps of other sizes or with respect to the diffusion part.

Minimum variance hedging

Idea: choose ϕ_t to minimize the mean square hedging error $E[\epsilon(\phi)^2]$

Mathematically: projection in $L^2(\Omega, \mathbb{Q})$ of payoff H_T on space of stochastic integrals w.r.t $S \Rightarrow$ **Kunita-Watanabe** decomposition

Elliott and Föllmer (2000), Benth et al (04), Corcuera et al (04), Arai (04, 05) : existence results, no explicit/computable representation except in trivial examples.

In a one-factor Markovian setting, the Kunita Watanabe decomposition can be explicitly computed: $\phi_t^0 = \Phi_{MV}(t, S_{t-})$ where

$$\Phi_{MV}(t, S) = \frac{S^2 \sigma^2(t, S) \frac{\partial C}{\partial S}(t, S) + \int \nu(dz) S(e^z - 1) C(t, S e^z) - C(t, S)}{S^2 \sigma(t, S)^2 + \int S^2 (e^z - 1)^2 \nu(dz)} \quad (30)$$

Tradeoff between hedging of diffusion or hedging of jumps.

Contrarily to Δ , well defined in all limiting cases: no diffusion (VG, NIG,...), single jump size,...

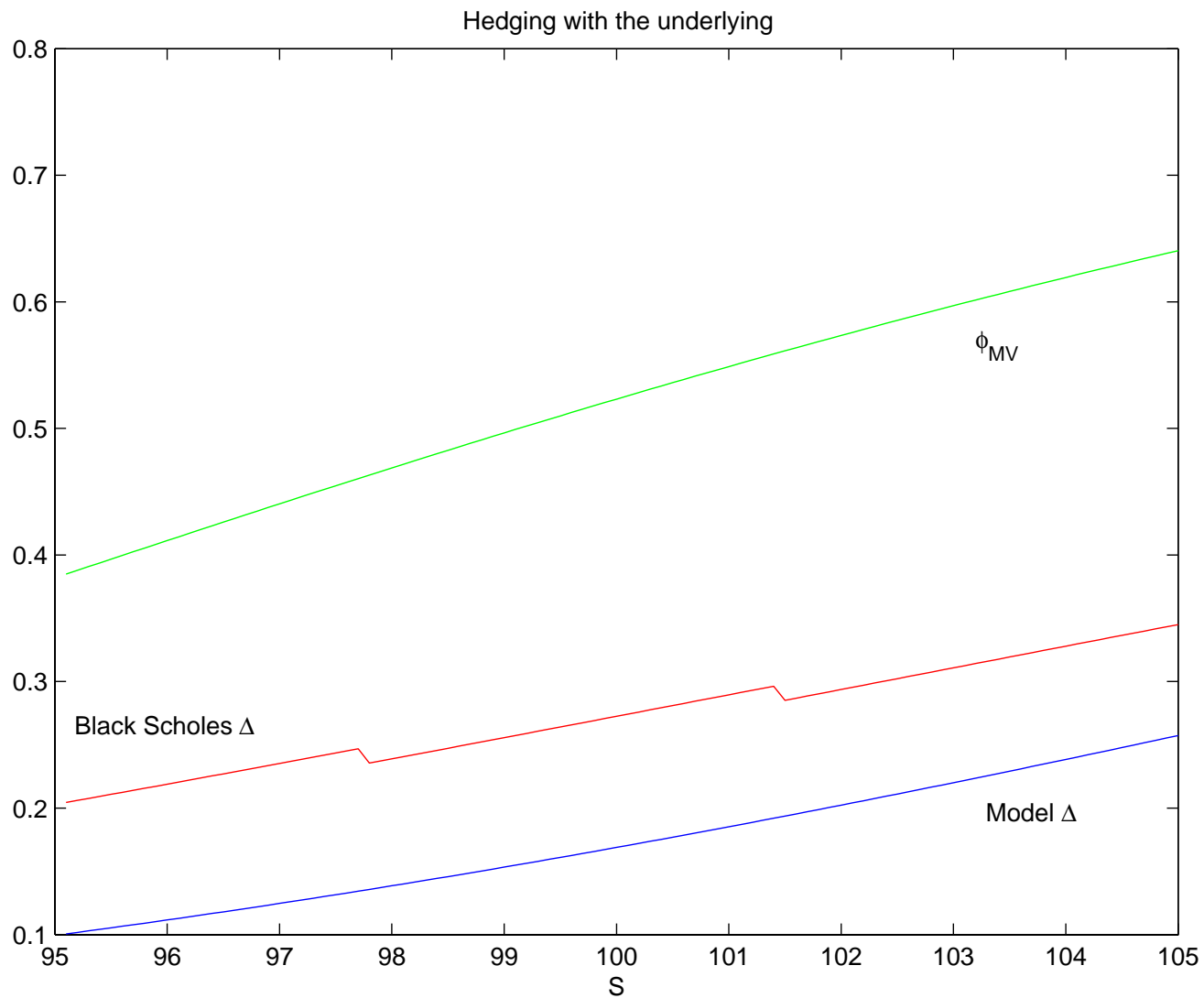


Figure 9: Model Δ , Black Scholes Δ and quadratic hedge ratio, in a Merton model.

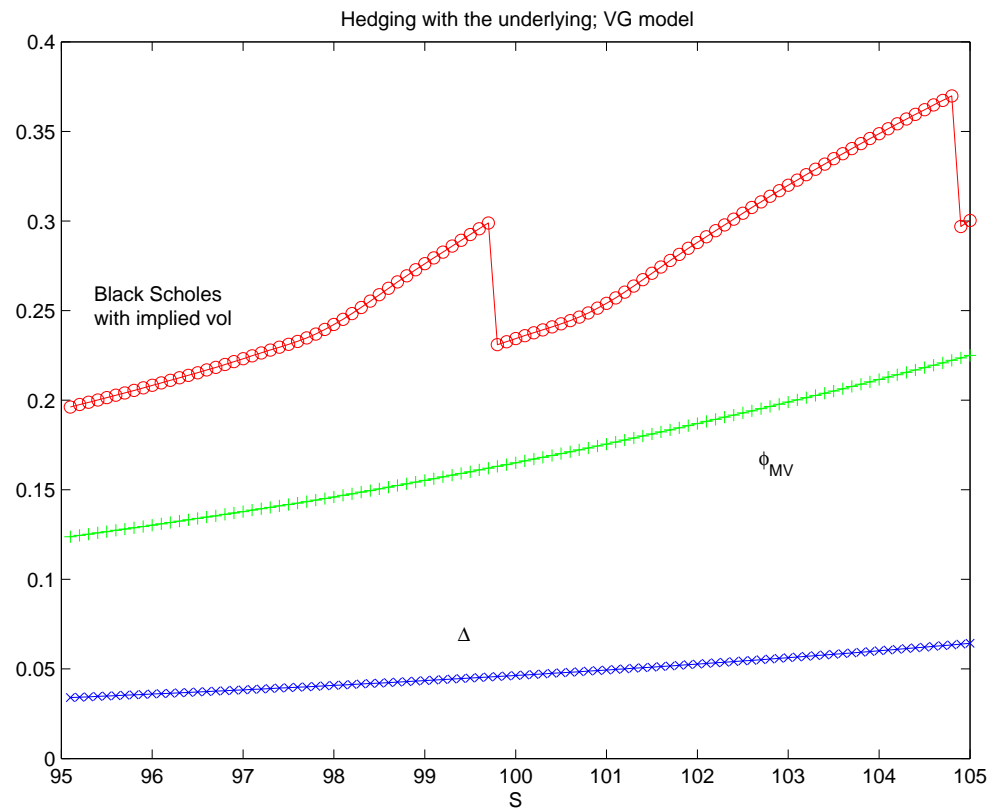


Figure 10: Hedging an out of the money call $K = 120$ with underlying in a Variance Gamma model.

		Case 1:	Case 2:
		bullish market	bearish market
Minimal	vari-	0.0091	0.0085
ance			
Δ		0.0387	0.0274
Unhedged		0.2209	0.2650
$\sqrt{P\&L^2}$			

Table 3: Delta neutral vs quadratic hedging strategies.

A stochastic volatility example: the Barndorff-Nielsen Shepard model

$$dY_t = \left(-l(\rho) - \frac{1}{2}\sigma_t^2\right)dt + \sigma_t dW_t + \rho dZ_t \quad (31)$$

$$d\sigma_t^2 = -\lambda\sigma_t^2 dt + dZ_t, \quad \sigma_0^2 > 0 \quad (32)$$

where $l(\theta) = \log E(e^{\theta Z_1})$, $\rho \leq 0$, $\lambda > 0$. Then

$$\phi_t = \frac{\sigma_{t-}^2 \frac{\partial C}{\partial X} + \frac{1}{X_{t-}} \int \nu(dz)(e^{\rho z} - 1)[C(t, X_{t-}e^{\rho z}, \sigma_{t-}^2 + z) - C(t, X_{t-}, \sigma_{t-}^2)]}{\sigma_{t-}^2 + \int (e^{\rho z} - 1)^2 \nu(dz)}.$$

When there are no jumps in the stock price ($\rho = 0$) the optimal hedging strategy is just delta-hedging: $\phi_t = \frac{\partial C}{\partial X}$; even though there are jumps in the option price, they cannot be hedged.

$\Delta - \Gamma$ hedge

$$dC_t = \frac{\partial C}{\partial S} dS_t + \frac{\partial C}{\partial t} dt + \frac{\sigma^2(t, S_t) S_t^2}{2} \frac{\partial^2 C}{\partial S^2} dt + C(t, S_t) - C(t, S_{t-}) - \frac{\partial C}{\partial S}(t, S_{t-}) \Delta S_t$$

If jump size not too large:

$$C(t, S_t) - C(t, S_{t-}) - \frac{\partial C}{\partial S}(t, S_{t-}) \Delta S_t \simeq \underbrace{\frac{1}{2} \frac{\partial^2 C}{\partial S^2}}_{\Gamma} |\Delta S_t|^2 + o(|\Delta S_t|^2) \quad (33)$$

Principal contribution of jumps is to increase “Gamma” risk. \Rightarrow
Neutralize first (Δ) and second (Γ) order sensitivity to S_t

- Neutralize Gamma by taking position in options:

$$\Gamma_t^0 + \sum_{i=1}^k \phi_t^i \Gamma_t^i = 0$$

- Neutralize residual delta by taking position in futures:

$$\phi_t^0 = -\Delta_t^0 - \sum_{i=1}^k \phi_t^i \Delta_t^i$$

Using the realized volatility (quadratic variation) $[S]_t$:

$$\begin{aligned}
 dC_t &= \frac{\partial C}{\partial S}(t, S_{t-})dS_t + \frac{\partial C}{\partial t}dt + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} d[S]_t & (34) \\
 &+ \underbrace{C(t, S_t) - C(t, S_{t-}) - \frac{\partial C}{\partial S}(t, S_{t-})\Delta S_t - \frac{1}{2} \frac{\partial^2 C}{\partial S^2} |\Delta S_t|^2}_{\simeq \partial^3 C / \partial S^3 (\Delta S_t)^3 / 6}
 \end{aligned}$$

For a $\Delta - \Gamma$ -neutral portfolio, the contribution of a jump ΔS_t to P&L $\simeq \partial^3 C / \partial S^3 (\Delta S_t)^3 / 6 + o(\Delta S_t)^3$, if the jump size is not too large.

\Rightarrow asymmetric behavior with respect to price jumps

	Historical model	Risk-neutral model
Drift	0.2	—
Volatility	0.1	0.1
Jump intensity	5	10
Mean jump size	-0.05	-0.2
Jump std. dev.	0.1	0.2

Table 4: Historical and risk-neutral parameters used for simulations: note the jump risk premium.

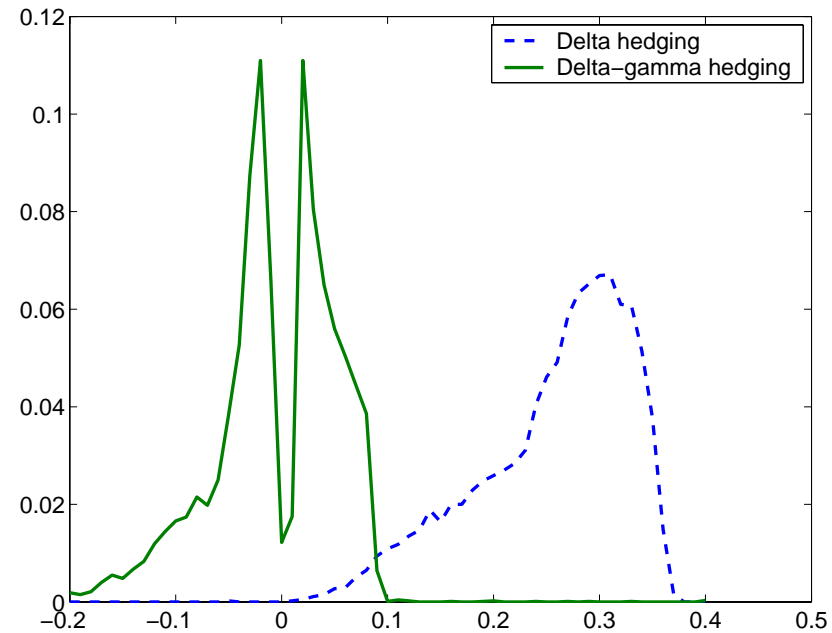


Figure 11: Empirical P&L distributions for delta and delta-gamma hedging strategies. Mean absolute hedging error equals to 0.2562 for delta-hedging and to 0.0462 for gamma-delta hedging.

Delta-Gamma hedging: remarks

Delta-Gamma hedge strongly reduces bias in P&L: centered P&L distribution

Robust to presence of jump risk premium

BUT:

Residual risk is still large

Can increase the loss in case of large jumps

Gamma is not well defined (blows up!) in many pure jump models

Mean-variance hedging with underlying + options

Proposition 3 *The variance-minimizing hedge $\phi_t = (\phi_t^0, \phi_t^1, \dots, \phi_t^n)$ for the contingent claim H_T with value $C(t, S_t)$ using the underlying S_t and liquid options $C_t^* = (C^i(t, S_t), i = 1..n)$ is given by*

$$\phi_t = \left(a a^t + \int_{\mathbb{R}} \nu(dz) b(z) b(z)^t \right)^{-1} \left(a_0 a + \int_{\mathbb{R}} \nu^Q(dz) b_0(z) b(z) \right) \quad (35)$$

where

$$a_0 = \sigma_t S_t \frac{\partial C(t, S_t)}{\partial S} \quad b_0(z) = C(t, e^z S_{t-}) - C(t, S_{t-})$$

$$a = \begin{pmatrix} \sigma_t S_t \\ \sigma_t S_t \frac{\partial C^*(t, S_t)}{\partial S} \end{pmatrix} \quad b(z) = \begin{pmatrix} (e^z - 1) S_{t-} \\ C^*(t, e^z S_{t-}) - C^*(t, S_{t-}) \end{pmatrix}$$

Case of a single jump size

If only one jump size x_0 is possible (for example, jump to default) then, denoting $\Delta S := (e^{x_0} - 1)S$ and $\Delta C(t, S) = C(t, Se^{x_0}) - C(t, S)$, we obtain

$$\phi_t^0 = \Phi^0(t, S_{t-}) \quad \phi_t^1 = \Phi^1(t, S_{t-})$$

$$\Phi^0(t, S) = \frac{(C^*(t, Se^{x_0}) - C^*(t, S)) \frac{\partial C(t, S)}{\partial S} - (C(t, Se^{x_0}) - C(t, S)) \frac{\partial C^*(t, S)}{\partial S}}{C^*(t, Se^{x_0}) - C^*(t, S) - S(e^{x_0} - 1) \frac{\partial C^*(t, S)}{\partial S}}$$

$$\Phi^1(t, S) = \frac{C(t, Se^{x_0}) - C(t, S) - S_{t-}(e^{x_0} - 1) \frac{\partial C(t, S)}{\partial S}}{C^*(t, Se^{x_0}) - C^*(t, S) - S_{t-}(e^{x_0} - 1) \frac{\partial C^*(t, S)}{\partial S}}$$

This leads to a perfect hedge $\epsilon(\phi) = 0$: one option + underlying allows to replicate any other option.

Remark: this replication strategy is **not** delta-neutral!

When the jump ΔS is small, $\Delta C(t, S) \approx \Delta S \frac{\partial C}{\partial S} + \frac{1}{2} \Delta S^2 \frac{\partial^2 C}{\partial S^2}$ and the above formula simplify to the delta-gamma neutral hedge ratios:

$$\phi_t^0 = \frac{\partial C(t, S)}{\partial S} - \frac{\partial C^*(t, S)}{\partial S} \left(\frac{\partial^2 C(t, S)}{\partial S^2} / \frac{\partial^2 C^*(t, S)}{\partial S^2} \right)$$

$$\phi_t = \frac{\partial^2 C(t, S)}{\partial S^2} / \frac{\partial^2 C^*(t, S)}{\partial S^2}$$

More generally, the variance-optimal hedge leads to a perfect hedge when number of options used = number of jump sizes.

In this case, the delta or delta-gamma neutral strategies lead to a > 0 residual risk.

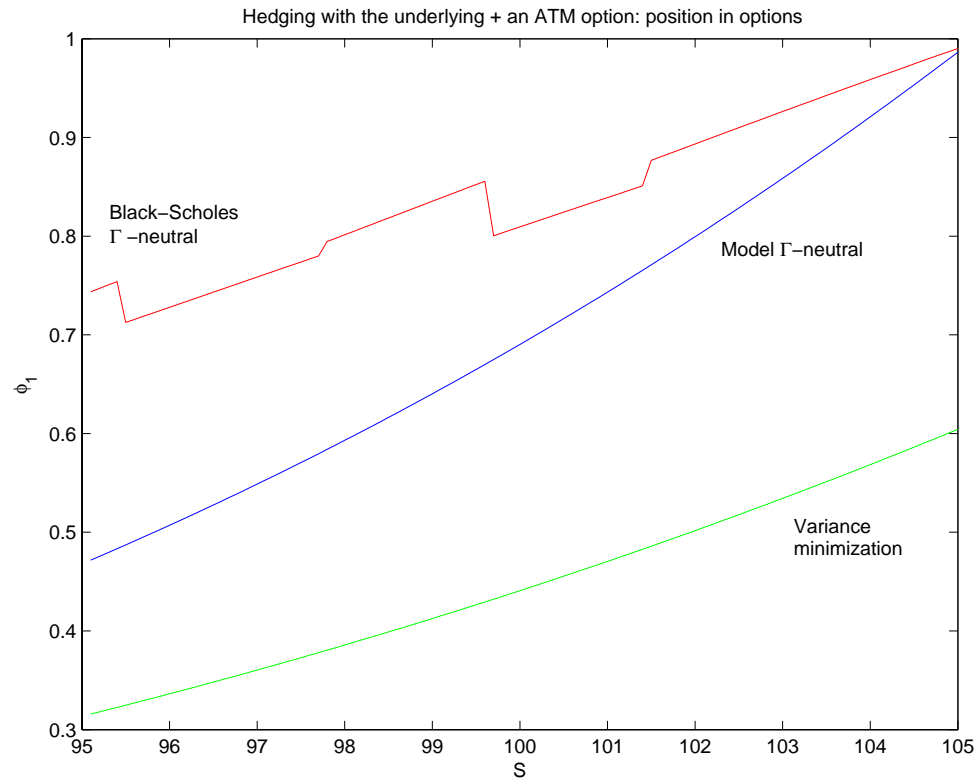


Figure 12: Hedging an out of the money call $K = 120$ with a liquid at-the-money call $K = S = 100 +$ futures in a Merton jump-diffusion model. Position in ATM call.

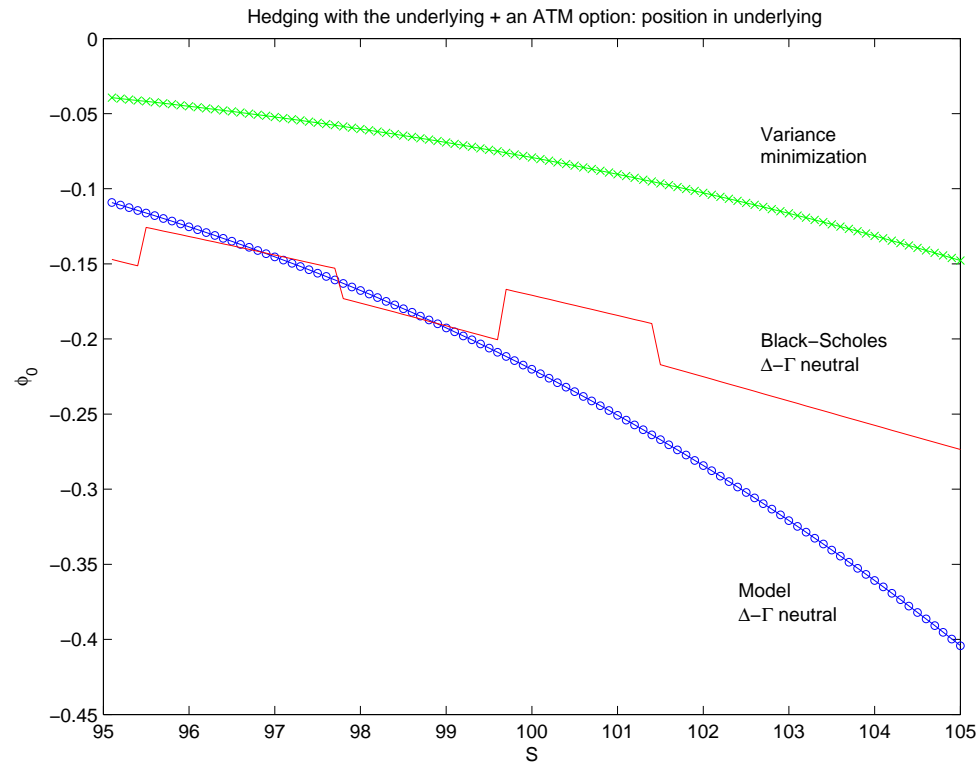


Figure 13: Hedging an out of the money call $K = 120$ with a liquid at-the-money call $K = S = 100$ + futures in a Merton jump-diffusion model. Position in underlying.

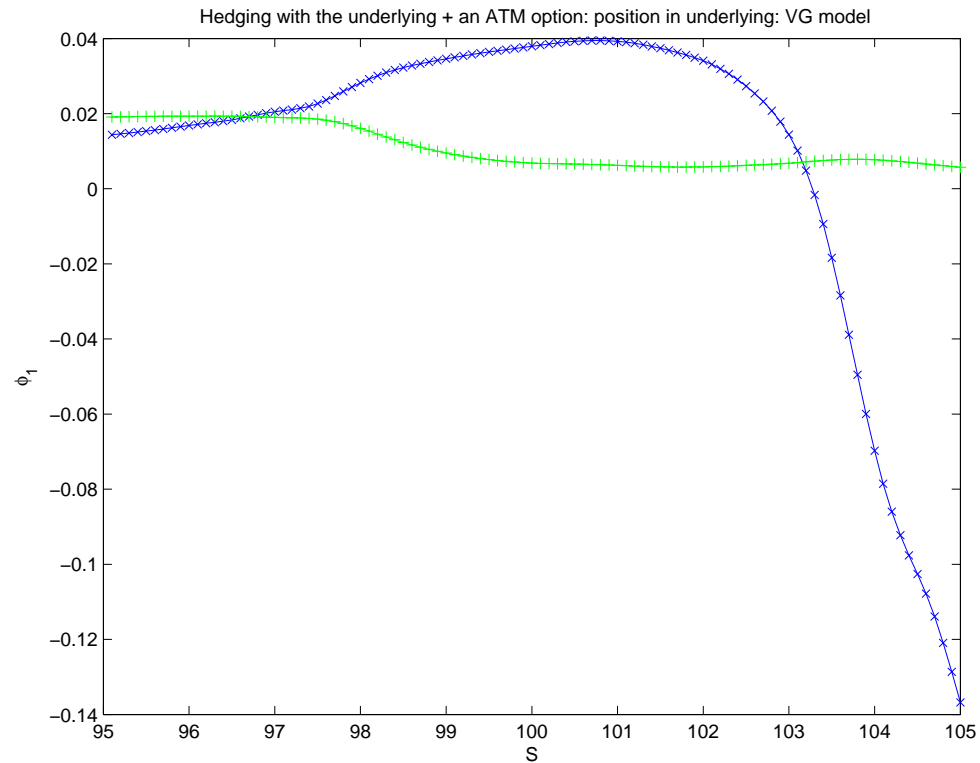


Figure 14: Hedging an out of the money call $K = 120$ with a liquid at-the-money call $K = S = 100$ + futures in a Variance Gamma model. Position in underlying.

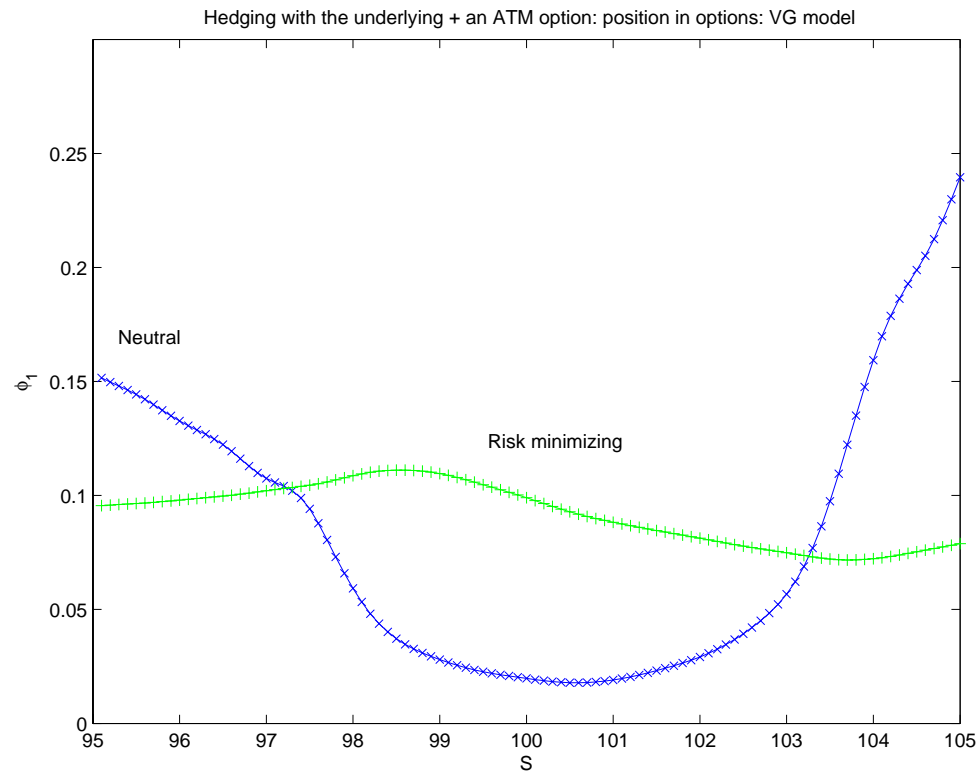


Figure 15: Hedging an out of the money call $K = 120$ with a liquid at-the-money call $K = S = 100$ + futures in a Variance Gamma model. Position in underlying.

Strategy	Fear of crash	Bullish market
Quadratic with Stock+option	0.0307	0.0072
Quadratic with Stock	0.2068	0.0305
Delta-neutral	0.2142	0.0388

Table 5: Mean squared $P&L$ of the optimal quadratic martingale hedging strategy with stock only, with one stock and one option, and of the delta hedging strategy.

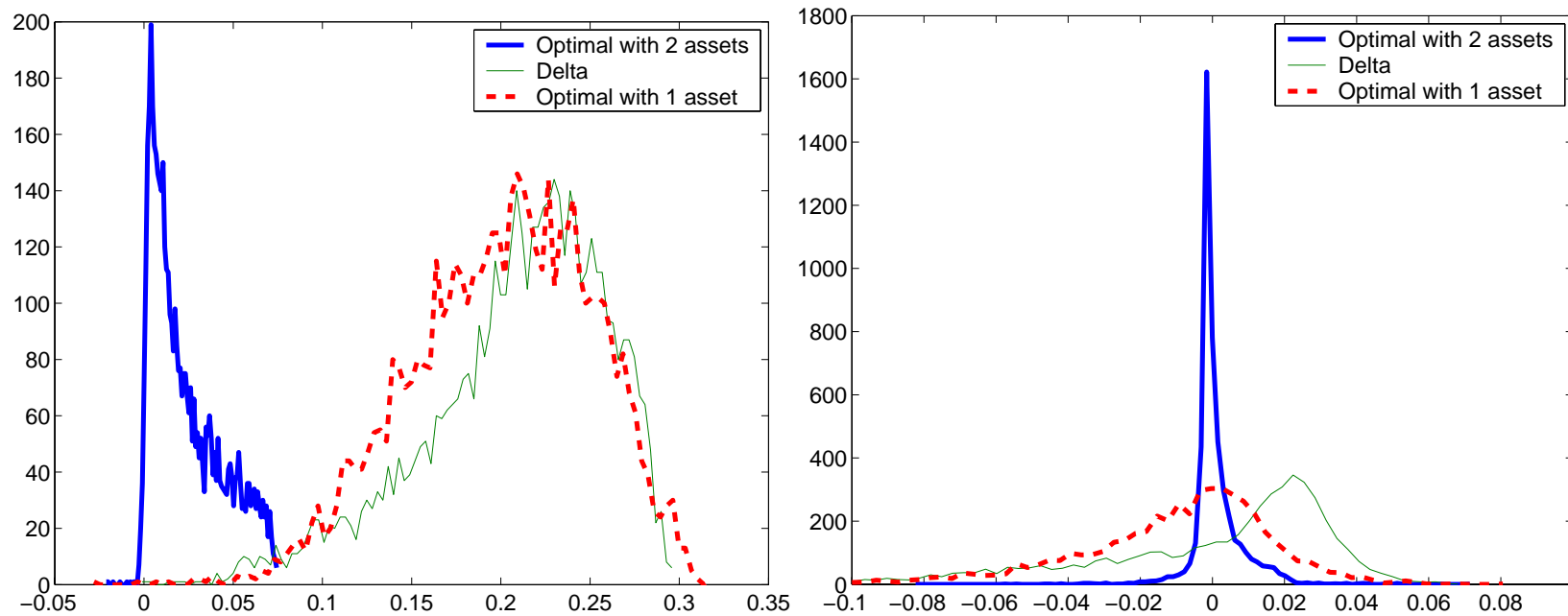


Figure 16: Minimal variance hedging strategy with stock + one option in a market with jump risk premium (left) and in a bullish market (right), compared to the delta hedging strategy and to variance minimizing strategy using underlying only.

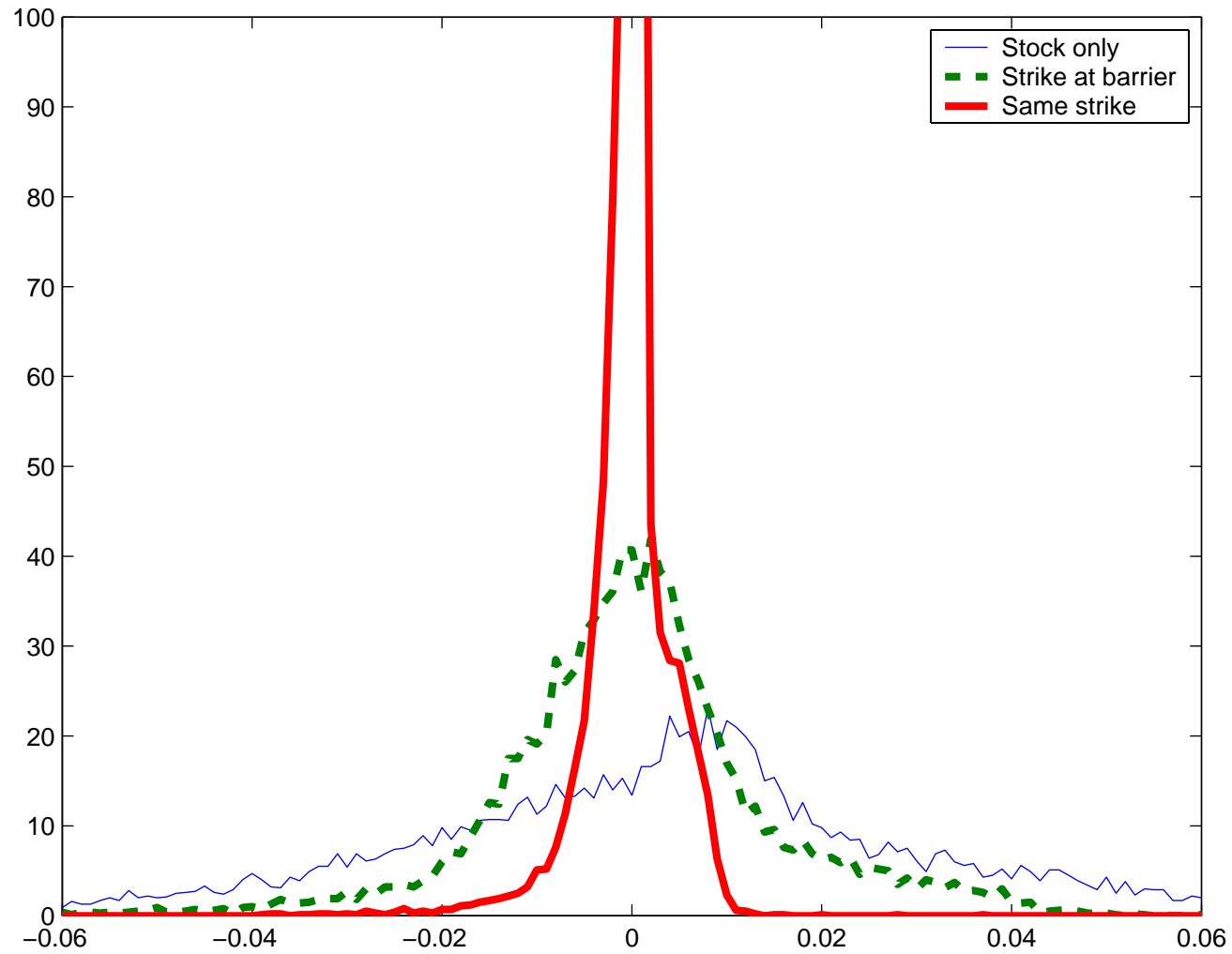


Figure 17: Residual hedging error for an up and out put with strike $K = 1$ and barrier $B = 1.2$.

CONCLUSION

- **Market incompleteness** is not a problem of a class of models: it is a market reality, leading to hedging errors. While complete market models ignore this problem, models with jumps allow to quantify the hedging errors.
- **Hedging \neq “replication”**: replication is a theoretical limiting case, while hedging is about *approximating* a payoff with other payoffs.
- **Hedge ratio \neq sensitivity!**
- Sensitivities (Δ, Γ) may be ill-defined/infinite in some pure jump models.
- Delta hedging can sometimes increase risk with respect to no hedging!
- Models with jumps capture Gamma risk in a meaningful way, even in continuous time.

- Hedging with options allows to center the P&L distribution, even in presence of strong jump risk premia.
- $\Delta - \Gamma$ hedge ratios are quite different from Black Scholes $\Delta - \Gamma$ hedges computed using the observed implied volatility skew!
- $\Delta - \Gamma$ **neutral** hedging always lead to a non-zero residual risk **even** in cases when replication is possible.
- Variance minimizing hedge ratios have the correct behavior in limiting cases (no jumps, small jump size, finite number of jump sizes,..), which is not the case for $\Delta - \Gamma$ neutral hedges.
- These strategies lead to different distributions residual P&L and **different sensitivities with respect to risk premia**.
- Hedge ratios can be pre-computed as functions of time/ underlying on a grid using finite-difference solvers for PIDEs.

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