# Continuous-Time Models and the Distribution of Daily Stock Returns

(Preliminary Version - November 2005)

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# INTRODUCTION

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History of Seeing Asset Returns as **i.i.d. Normal** is Long

Many **Small Uncorrelated Shocks** → **Gaussian** Distribution

Mixture-of Distribution (**MDH**) Theory: Mixed Gaussian

Empirically: What is the **Mixing Variable**?

Potential Complication: Large Moves (**Jumps**)

Also **Correlation**: Mixing Variable ↔ Return Innovation

# INTRODUCTION

(G)ARCH and Stochastic Volatility Models at Daily Level remain Workhorse

A Decade of HF Data for Cross-Section of Actively traded Assets now available

Issue: HF Data not been harnessed for direct Volatility Modeling at Daily Level

Progress using Summary Statistics Extracted from HF Data, e.g., Daily Range

We focus on the recent **Realized Volatility Measures**

# Main Inspiration and Objective

Relationship b/w standard Daily ARCH type Models and Realized Volatility Measures is Underdeveloped, both Theoretically and Empirically

Explore Characteristics of HF Return/Volatility Processes and their Implications for Daily Return Distributions

Show Arbitrage-Free Jump-Diffusion Setting provides a Flexible Setting for Exploring and Rationalizing Properties of Daily Asset Return Data

Real-Time Evolution of Quantities in Realized Volatility Literature related to Daily Return Characteristics

"Explain" **Conditional Fat-Tails in Standardized Return Innovations**,
**Asymmetric Return-Volatility Relation** and
**Extreme Outliers** from Underlying Features

Provide Sequence of **Nonparametric Test Procedures** and **Transformations** to Assess Strength of various Features under minimal auxiliary assumptions

# Further Objectives

Introduce Concept of **Generalized Volatility Signature Plots**

Informal Diagnostic Tool as Microstructure Noise-Robust Jump Indicator

Novel **Sequential Procedure for Jump Detection**

Idea is to Identify **Timing of Jump**

Able to **Detect Multiple Jumps** per Trading Day

After Jump Extraction, Empirical **Test for Gaussian Standardized Returns**

Data: HF Returns for **30 Individual Dow-Jones Industrial Average Stocks**

Also Explores **Financial-Time Return Distribution** (Leverage Effect undone)

# Theoretical Setting

Continuous Quoting/Trading in Markets w/ Instantaneous Information Transmission motivates No-Arbitrage Continuous-Time Modeling

$$dp(t) \;=\; \mu(t)\,dt \;+\; \sigma(t)\,dW(t) \;+\; \kappa(t)\,dq(t), \quad 0 \leq t \leq T,$$

$\mu(t)$ continuous locally bounded variation process (set ≡ 0 for simplicity)
$\sigma(t)$ strictly positive and caglad
$dq(t)$ counting process - equal to 1 for jump at time t, zero otherwise
$\kappa(t)$ indicates jump size if jump occurs

Associated **Quadratic Variation** Process

$$[r,r]_t \;=\; \int_0^t \sigma^2(s)\,ds \;+\; \sum_{0<s\leq t} \kappa^2(s)\,.$$

# Realized Volatility

$r_{t,\Delta} \equiv p(t) - p(t-\Delta)$ Discrete Δ-period (intraday) returns

**Realized Volatility** [Realized (Quadratic) Variation]

$$RV_{t+1}(\Delta) \equiv \sum_{j=1}^{1/\Delta} r^2_{t+j\cdot\Delta,\Delta}$$

**Convergence Uniformly in Probability**

$$RV_{t+1}(\Delta) \rightarrow [r,r]_{t+1} - [r,r]_t = \int_t^{t+1} \sigma^2(s)\,ds + \sum_{t<s\leq t+1} \kappa^2(s)$$

Andersen and Bollerslev (1998), ABDL (2001, 2003)
Barndorff-Nielsen and Shephard (2002a, 2002b)
Comte and Renault (1998)

# Pure Diffusive No Leverage Case

Quadratic variation the Natural **Return Variability Measure**

$$r(t) \mid \sigma\{[\mu(\tau), \sigma(\tau)]_{0 \le \tau \le t}\} \;\sim\; N\left(\int_0^t \mu(s)\,ds,\; \int_0^t \sigma^2(s)\,ds\right)$$

Standardized Return Implications $(\mu(t) \equiv 0)$, let

$$V(t) \;=\; E\left[\int_0^t \sigma^2(s)\,ds) \mid \mathscr{F}(0)\right] \;.$$

**Conditional Fat-Tailedness** $\quad r(t) \cdot V^{-1/2}(t) \;\nsim\; N(0,1)$

but,

$$r(t) \;\cdot\; \left[\int_0^t \sigma^2(s)\,ds\right]^{-1/2} \;\sim\; i.i.d.\; N(0,1)$$

# Return Sampling Considerations

In Principle, Sample as often as Possible, but **Market Microstructure Frictions** induce Violation of Semi-MG Assumption at the Highest Sampling Frequencies

Large Literature detailing problem and suggesting remedies

Aït-Sahalia, Mykland and Zhang (2005), ABDL (2000, 2003), Bandi and Russell (2004a,b), BNHLS (2005), Bollen and Inder (2002), Corsi, Zumbach, Müller and Dacorogna (2001), Hansen and Lunde (2006), Oomen (2004), Zhang, Aït-Sahalia and Mykland (2005), Zhou (1996)

We Sample less frequently to strike "sensible" balance b/w Information Accumulation and Microstructure Distortions - but Accept Errors Remain!

Approximate Semi-Martingale properties of individual Stock Returns sampled at **5-Minutes using Last Mid-Quote** so **78 Intraday Returns per Day** from TAQ over **01/98-12/02** yielding **1,255** Trading Days w/ HL (2006) cleaned HF Returns

# Test for Pure Diffusion without Leverage

Results Motivate a direct Test for whether the Daily Returns are Consistent with an Underlying Pure Diffusive Process without Leverage Effects

Compute Realized Volatility from HF data and Use for Standardization of Trading Day Return Series.   Then Test if RV-Standardized Returns are iid N(0,1) !

Such Realized-Volatility-Standardized Daily Returns are indeed much closer to Gaussian than raw Daily Returns or even GARCH Residuals
[ ABDL (2000, 2001), Andersen, Bollerslev, Diebold and Ebens (2001) ]

However, when Subjected to Powerful Tests for iid N(0,1) the Rejections are typically Overwhelmingly Strong

This Stem from **Failure of Null Hypothesis** or **Measurement Errors in Realized Volatility** implying poor proxies for the Integrated Variance Series

# The Impact of Jumps

Results above **Not Valid** if there are **Discontinuities in Sample Path**

Existence of **Nonparametric Jump Identification and Extraction Procedures** suggest **Gaussianity may be restored**

Define for $\mu_1 \equiv \sqrt{(2/\pi)}$ the **Standardized Realized Bipower Variation Measure**

$$BV_{t+1}(\Delta) \equiv \mu_1^{-2} \sum_{j=2}^{1/\Delta} | r_{t+j\cdot\Delta,\Delta} | \, | r_{t+(j-1)\cdot\Delta,\Delta} |$$

It follows that for $\Delta \rightarrow 0$ [Barndorff-Nielsen and Shephard (2004, 2005)]

$$BV_{t+1}(\Delta) \rightarrow \int_t^{t+1} \sigma^2(s)\, ds$$

# The Impact of Jumps (continued)

The Ability to Measure the Integrated Variance, even in Presence of Jumps, allows for Construction of Jump Identification and Extraction Procedures

BNS **Asymptotic Distribution Theory** enable **Jump Tests**, e.g.,

$$RV_{t+1}(\Delta) - BV_{t+1}(\Delta) \rightarrow \sum_{t<s\leq t+1} \kappa^2(s)$$

Define $$J_{t+1}(\Delta) \equiv \max[\, RV_{t+1}(\Delta) - BV_{t+1}(\Delta)\, ,\, 0\,]$$

Asymptotically Valid Test for $J_{t+1}(\Delta)$ Significant - we use **p-value 1%** (0.1, 5%)

**Practical Jump Identification Procedures** by ABDobrev (2005), BNS (2005), Huang and Tauchen (2005) - Use **Transform Z of above J-Statistic**

We employ another Variant here w/ specific Objectives in mind

# Generalized Volatility Signature Plots

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The HF Returns for Individual Stocks may be quite “Noisy”

Is Jump Detection Reliable - Can we Check if it is Necessary?

Informal Robustness Check: **Generalized Volatility Signature Plots**

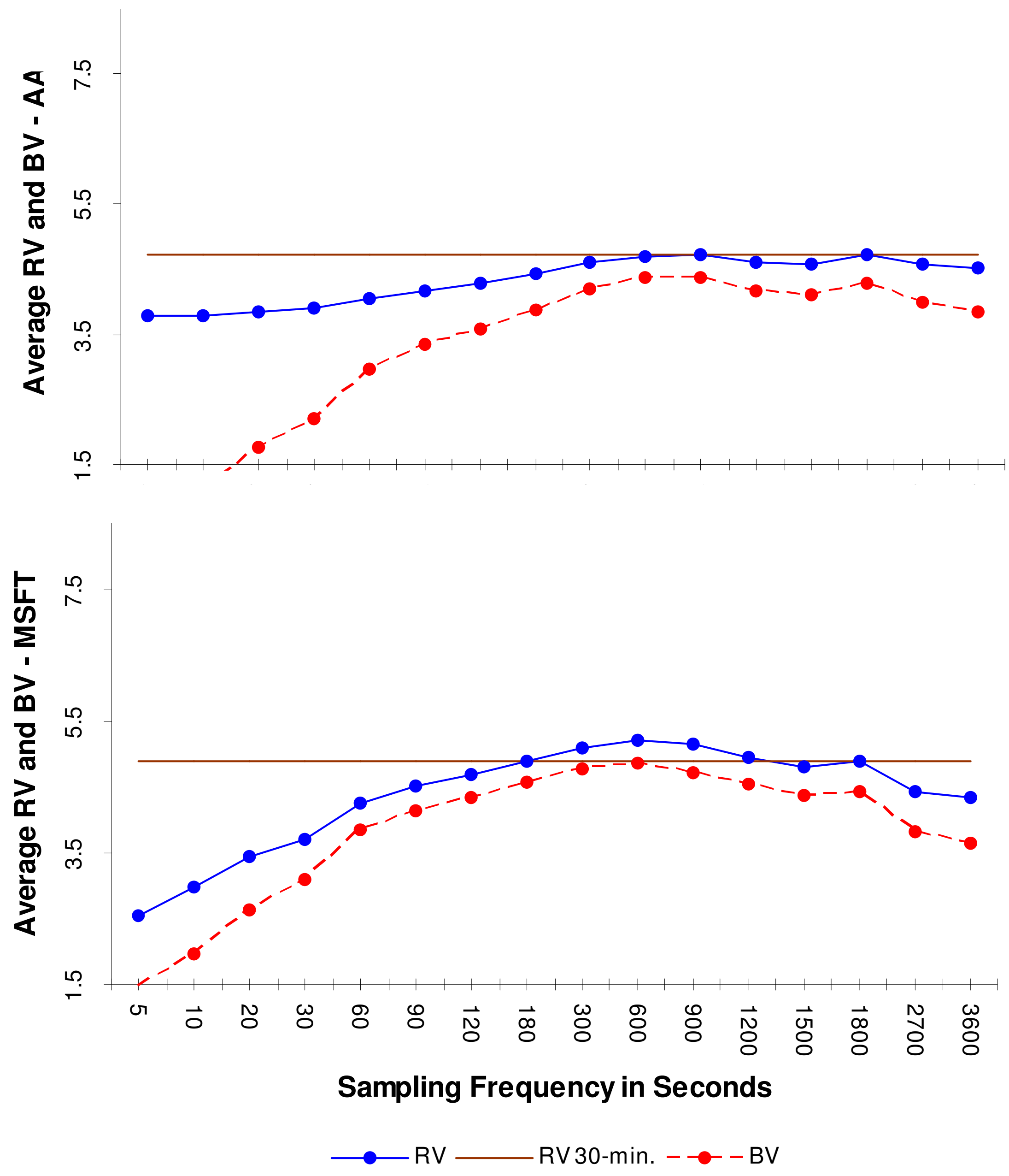
Plot **Average Realized Volatility Style Measure** against **Sampling Frequency**

In terms of rendering the Measures more Robust to i.i.d. Noise, we may use

- the $AC_1$ -Type Correction to RV [Hansen and Lunde (2006)]
- the Skip-One Approach for BV [Huang and Tauchen (2005)]

The Discrepancies in the RV and BV displays should Indicate the “Average” Contribution of Jumps to QV per Trading Day.

**Figures 1A and 1B: Bipower Variation Signature Plots**



**Figures 2A and 2B: Robustified Signature Plots**

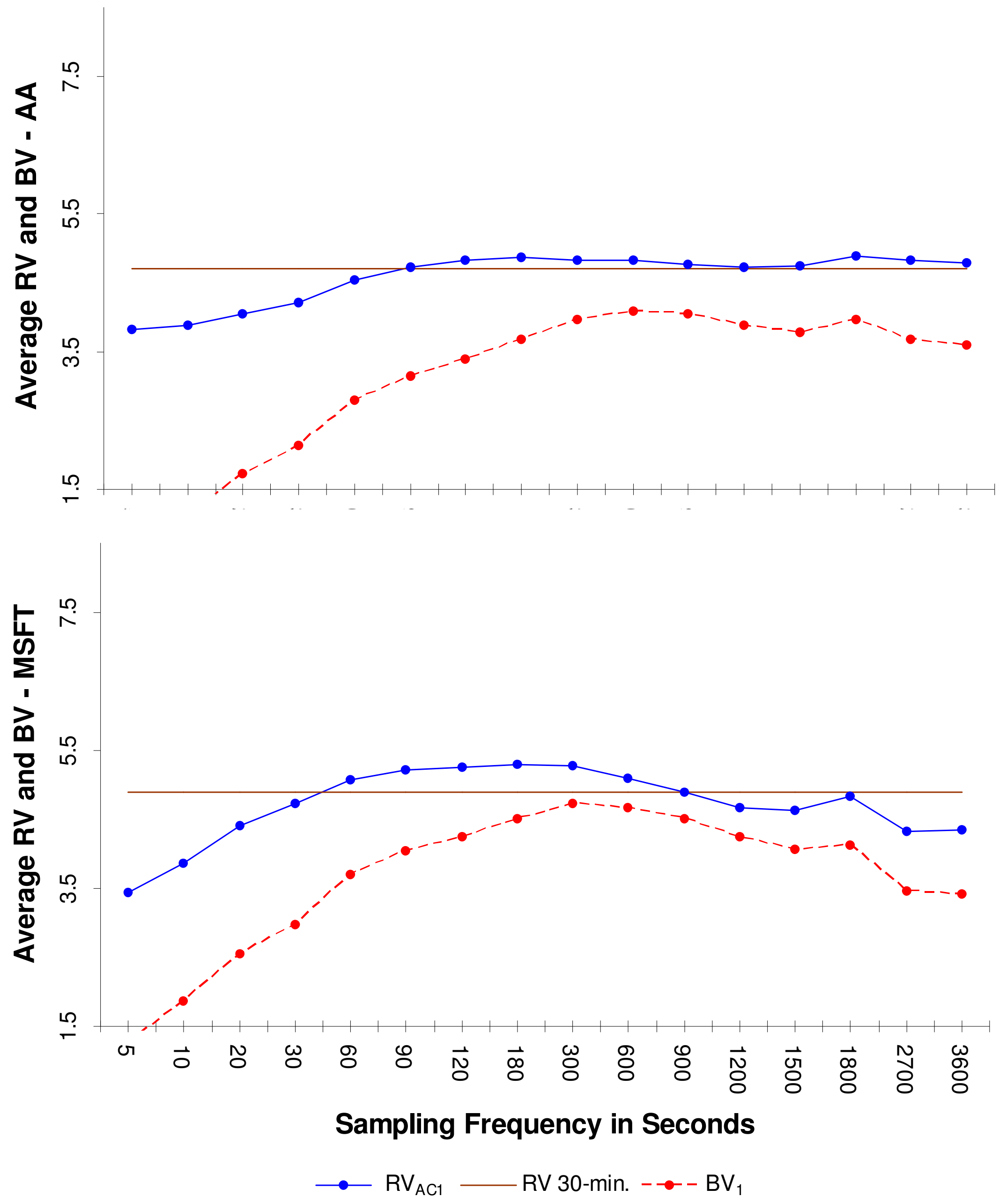


Figure 1: Generalized volatility signature plots for AA-INTC stocks

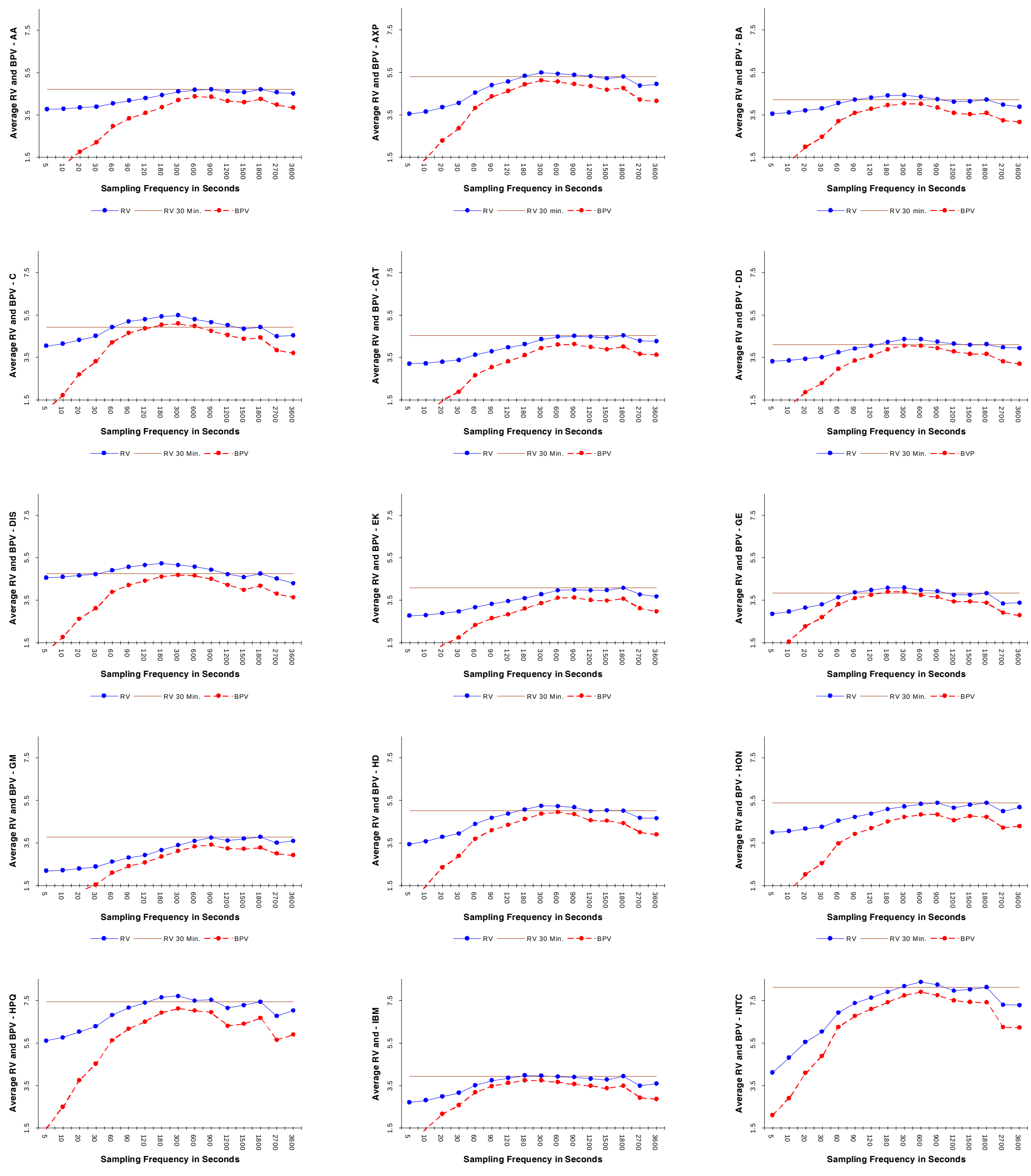
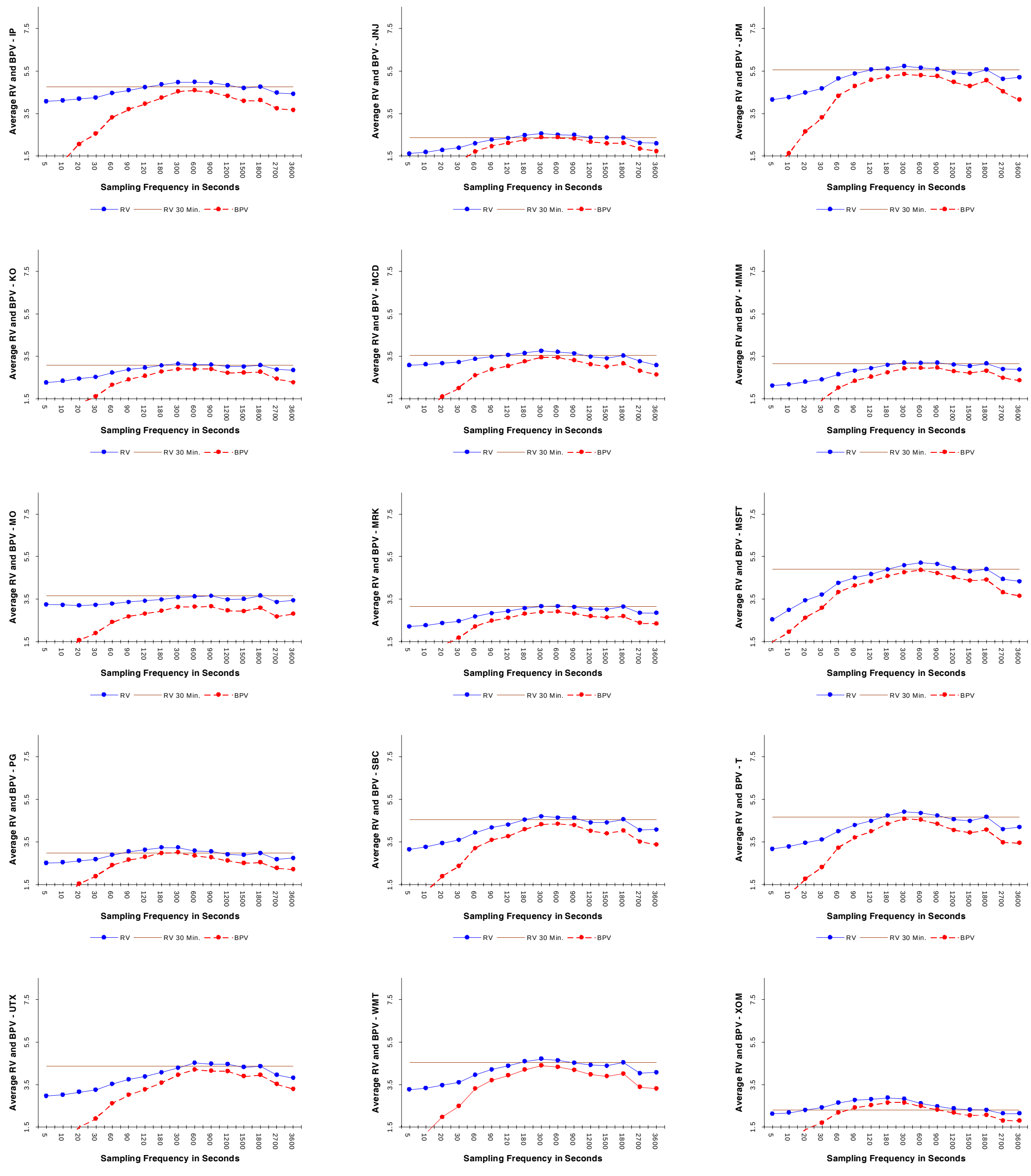


Figure 1 cont.: Generalized volatility signature plots for IP-XOM stocks



# Identifying Jump Components in Daily Returns

**First (Simple) Method -** Sign of Largest 5-Minute Return

$$\tilde{\kappa}_t \;=\; sgn[\,r_{t,k} : |r_{t,k}| \;=\; \max_{j=1,\ldots,M} |r_{t,k}|\,]\;\sqrt{J_t}$$

**Second Method** - Sequential Jump Testing

$$JS_{t,1} \;=\; \mathbb{I}(Z_t > \Phi_{1-\alpha}) \cdot [\max_{j \in (1,\ldots,M)} r_{t,j}^2 - \frac{1}{M-1} \sum_{k=1,\,k \neq j}^{M} r_{t,k}^2\,]$$

$$\hat{\kappa}_{t,1} \;=\; r_{t,j_1}$$

.....

$$JS_{t,i} = \mathbb{I}(Z_t > \Phi_{1-\alpha}) \cdot [\max_{j \in (1,\ldots,M) \setminus (j_1,\ldots,j_{i-1})} r_{t,j_i}^2 - \frac{1}{M-J} \sum_{k=1,\,k \neq (j_1,\ldots,j_i)}^{M} r_{t,k}^2\,]$$

$$\hat{\kappa}_{t,i} \;=\; r_{t,j_i}$$

**‘Estimated’ Jump Contribution to RV w/ Sequential Jump Detection**

$$JS_t = \sum_{i=1}^{J} JS_{t,i}$$

**Diffusive Volatility Contribution to RV after Sequential Jump Detection**

$$CVS_t = RV_t - JS_t$$

Let $r_t = r_{t,1} \equiv p(t) - p(t\text{-}1)$ Trading Day Return

**Jump-Adjusted Return after Sequential (Simple) Jump Detection**

$$\hat{r}_t = r_t - \sum_{i=1}^{J} \hat{\kappa}_{t,i} \qquad \left( \tilde{r}_t = r_t - \sum_{i=1}^{J} \tilde{\kappa}_t \right)$$

**Test if Jump-Adjusted RV-Standardized Trading Day Returns are Gaussian**

Table 1: Jumps - Simple Method

| | Mean duration | Rel. jump contribution $JV_t/RV_t$ | Mean size of jump component (x10,000) | Mean size of actual jumps (%) |
|---|---|---|---|---|
| Mean across stocks | 6.3201 | 0.0476 | 1.2119 | 0.9812 |
| Std. dev. across stocks | 1.6068 | 0.0133 | 0.3283 | 0.1233 |
| Min. across stocks | 4.1325 | 0.0256 | 0.6247 | 0.7352 |
| Max. across stocks | 10.0976 | 0.0746 | 2.0825 | 1.3121 |

Note: The table reports the mean, standard deviation, minimum, and maximum over all 30 DJIA stocks for the mean durations between jumps, relative jump contribution to realized volatility, mean size of jump component (x10,000) on days of non-zero jumps, and mean size in percent of the square-root jump component (i.e. the absolute value of the actual jumps themselves) on days of non-zero jumps. For details, see Table **??** of the appendix.

Table 2: Simple and Sequential Jump Correlations

| | Correlation | RMSE | Theil's U |
|---|---|---|---|
| Mean across stocks | 0.9450 | 0.0062 | 0.2999 |
| Std. dev. across stocks | 0.0332 | 0.0033 | 0.1036 |
| Min. across stocks | 0.8722 | 0.0030 | 0.1086 |
| Max. across stocks | 0.9945 | 0.0200 | 0.5508 |

Note: The table reports the mean, standard deviation, minimum, and maximum over all 30 DJIA stocks for the correlation, root mean squared error (RMSE) and Theil's U statistic for the two jump component series (using simple and sequential method). Observations where both series are zero have been removed. For details, see Table **??** of the appendix.

Table 3: Jumps - Sequential Method

| | Rel. jump contribution $JVS_t/RV_t$ | Mean size of jump component (x10,000) | Mean size of actual jumps (%) |
|---|---|---|---|
| Mean across stocks | 0.0373 | 1.0394 | 0.9282 |
| Std. dev. across stocks | 0.0101 | 0.3050 | 0.1177 |
| Min. across stocks | 0.0212 | 0.5065 | 0.6828 |
| Max. across stocks | 0.0575 | 1.8309 | 1.2464 |

Note: The table reports the mean, standard deviation, minimum, and maximum over all 30 DJIA stocks for the relative jump contribution to realized volatility, mean size of jump component (x10,000) on days of non-zero jumps, and mean size in percent of the absolute value of the actual jumps. For details, see Table **??** of the appendix.

# The Impact of Leverage Effects

Results above **Not Valid** when **Return and Volatility Innovations Correlated**

"**Time Change Theorem for Continuous Local Martingales**" suggests **Gaussianity may be restored** [Dambis (1965), Dubins and Schwartz (1965)]

We must Sample Returns in **Event** or **Financial Time** as given by, say, **τ***

Define $0 = t_0,\ t_1,\ t_2,\ \ldots,\ t_k, \ldots$, as Calendar Time points satisfying

$$t_k \;=\; inf_{t>0}\ ([r,r]_t - [r,r]_{t_{k-1}} \;>\; \tau^*), \qquad k \;=\; 0, 1, \ldots$$

Define **Financial Time Returns** as

$$R_k \;\equiv\; p(t_k) - p(t_{k-1}), \qquad k \;=\; 0,\ 1,\ 2,\ \ldots\ ,$$

Then $\qquad R_k \,/\, \sqrt{\tau^*} \quad \sim \quad i.i.d.\ N(0,1)\ , \qquad k = 0,\ 1,\ 2,\ \ldots\ .$

# Preliminary Leverage and Volatility Feedback Analysis

Individual Stocks Display Less Asymmetric Return-Volatility Relation than Index
Are there any Indication of their Presence and Impact?

For all 30 DJIA Stocks, Compute and Display HF Correlations,

Corr( | $r_{t,j}$ |, $r_{t,j+k}$ ) ,   for k = -K, ... , -1, 0 , 1, ... , K   (K=30)

This is Negative for k<0 if there is a Significant "**Leverage Effect**"
This is Positive for k>0 if there is a Significant "**Volatility Feedback Effect**"
Difference b/w them speaks to General Asymmetric Relationship

More Formal Test from (i= -2, ... K and i= 2,...,K respectively) differences
(w/ $r_j$ denoting J= MT- 2K + 1 demeaned 5-minute Return series)

$$\frac{1}{K-2} \sum_{|i|=2}^{K-1} \frac{1}{J} \sum_{j=0}^{J} \left( |r_{j+K}| \; r_{j+K-i} \right)$$

Table 4: Leverage and Volatility Feedback Estimates

| | Leverage effect | Feedback feedback | $p$-value |
|---|---|---|---|
| Mean across stocks | -0.0166 | 0.0076 | 0.1137 |
| Std. dev. across stocks | 0.0151 | 0.0087 | 0.1877 |
| Min. across stocks | -0.0560 | -0.0155 | 0.0000 |
| Max. across stocks | 0.0053 | 0.0226 | 0.7160 |
| Significance at 10% level | 22 | 13 | — |
| Significance at 5% level | 9 | 10 | — |
| Significance at 1% level | 6 | 5 | — |

Note: The table reports the mean, standard deviation, minimum, and maximum over all 30 DJIA stocks for the high-frequency leverage and volatility feedback estimates and $p$-values for the test for differences in leverage and volatility feedback based on Newey-West variance-covariances. The last three rows report the number of stocks (out of 30) for which the leverage resp. volatility feedback effects are significant at 10%, 5%, and 1% level. For details, see Table **??** of the appendix.

Table A5: High-frequency leverage and volatility feedback estimates

| Ticker | Leverage | Feedback | $p$-value |
|---|---|---|---|
| AA | 0.0053 (0.0013) | 0.0100 (0.0025) | 0.600 |
| AXP | -0.0306$^c$ (-0.0059) | 0.0068 (0.0017) | 0.001 |
| BA | -0.0104 (-0.0026) | -0.0069 (-0.0011) | 0.716 |
| C | -0.0025$^a$ (-0.0049) | 0.0109 (0.0024) | 0.020 |
| CAT | 0.0029 (0.0009) | 0.0088 (0.0023) | 0.481 |
| DD | -0.0119$^b$ (-0.0032) | 0.0140$^b$ (0.0038) | 0.001 |
| DIS | 0.0050 (0.0007) | 0.0217$^c$ (0.0046) | 0.154 |
| EK | -0.0121 (-0.0031) | -0.0006 (-0.0001) | 0.316 |
| GE | -0.0222$^c$ (-0.0058) | 0.0045 (0.0015) | 0.004 |
| GM | -0.0192$^c$ (-0.0060) | -0.0010 (0.0003) | 0.038 |
| HD | -0.0357$^c$ (-0.0069) | 0.0016 (0.0006) | 0.005 |
| HON | -0.0461$^c$ (-0.0092) | -0.0098 (-0.0016) | 0.019 |
| HPQ | -0.0192$^a$ (-0.0026) | 0.0163$^a$ (0.0029) | 0.034 |
| IBM | -0.0253$^c$ (-0.0068) | 0.0102$^b$ (0.0025) | 0.000 |
| INTC | -0.0560$^c$ (-0.0075) | 0.0098 (0.0058) | 0.000 |
| IP | -0.0097$^a$ (-0.0019) | 0.0095$^a$ (0.0020) | 0.047 |
| JNJ | -0.0021 (-0.0011) | 0.0139$^c$ (0.0058) | 0.010 |
| JPM | -0.0046 (-0.0008) | 0.0226$^a$ (0.0045) | 0.145 |
| KO | -0.0064$^a$ (-0.0022) | 0.0051 (0.0023) | 0.041 |
| MCD | -0.0184$^b$ (-0.0051) | 0.0065 (0.0019) | 0.009 |
| MMM | -0.0051 (-0.0015) | 0.0041 (0.0016) | 0.162 |
| MO | -0.0294$^c$ (-0.0082) | -0.0155 (-0.0041) | 0.214 |
| MRK | -0.0083$^a$ (-0.0029) | 0.0162$^c$ (0.0047) | 0.001 |
| MSFT | -0.0294$^c$ (-0.0064) | 0.0103 (0.0026) | 0.000 |
| PG | -0.0145$^c$ (-0.0046) | -0.0044 (-0.0012) | 0.227 |
| SBC | -0.0199$^b$ (-0.0045) | 0.0139$^b$ (0.0030) | 0.001 |
| T | 0.0039 (0.0007) | 0.0163$^b$ (0.0039) | 0.279 |
| UTX | -0.0309$^c$ (-0.0075) | 0.0078 (0.0022) | 0.000 |
| WMT | -0.0364$^c$ (-0.0083) | 0.0165$^b$ (0.0039) | 0.000 |
| XOM | -0.0121$^a$ (-0.0046) | 0.0092$^a$ (0.0037) | 0.000 |
| SP500 | -0.0126$^c$ (-0.0081) | 0.0063$^a$ (0.0042) | 0.000 |

Note: The table reports the high-frequency leverage and volatility feedback estimates and $p$-values for the test for differences in leverage and volatility feedback based on Newey-West variance-covariances. The numbers in parentheses are the (average) cross-correlations, and the given leverage and feedback estimates are the (average) cross-covariances multiplied by $10^5$. The superscripts $a$, $b$, and $c$ refer to significance at the 10%, 5%, and 1% level, respectively.

Figure 3: High-frequency leverage and volatility feedback effects, stocks AA-INTC

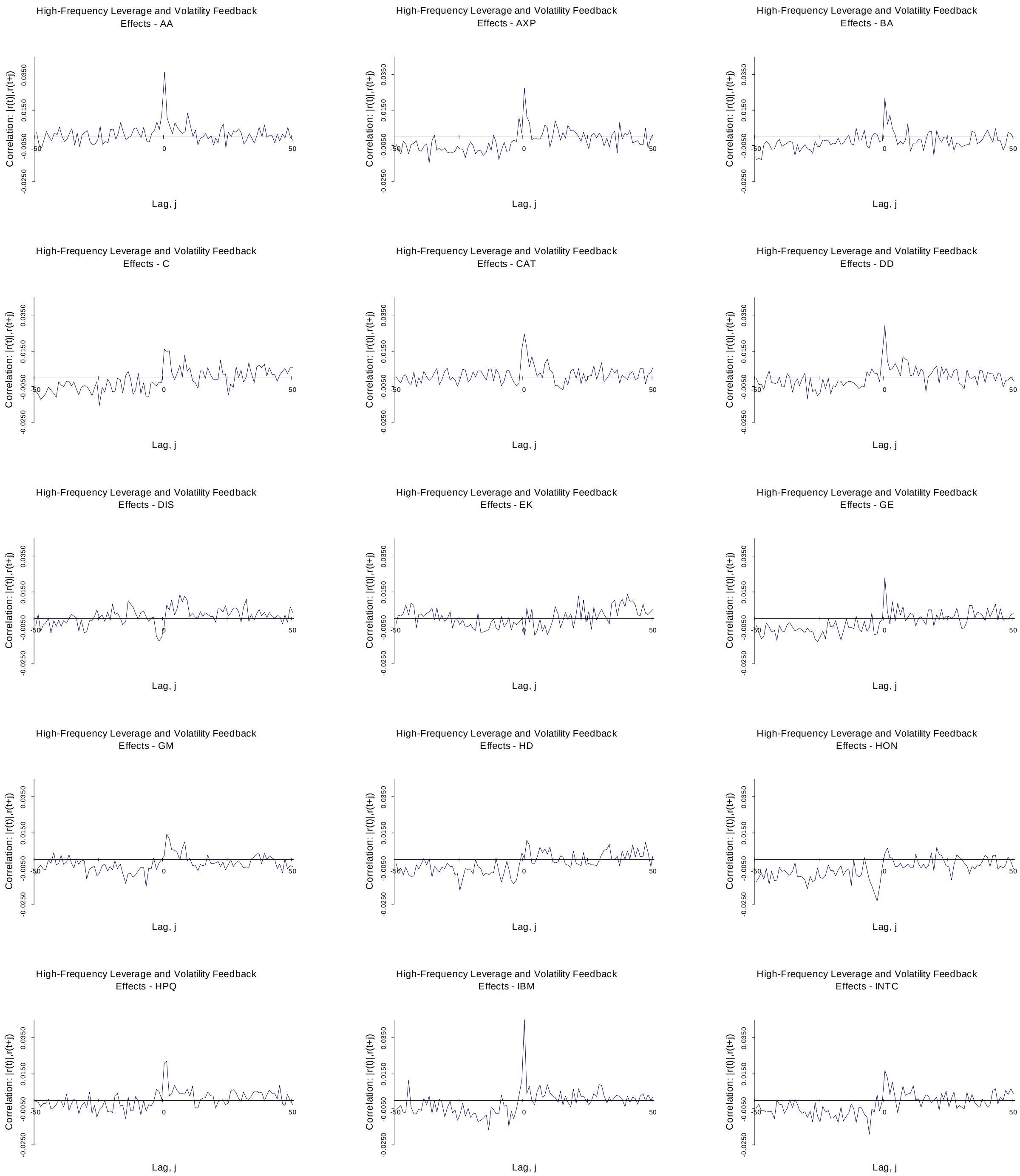
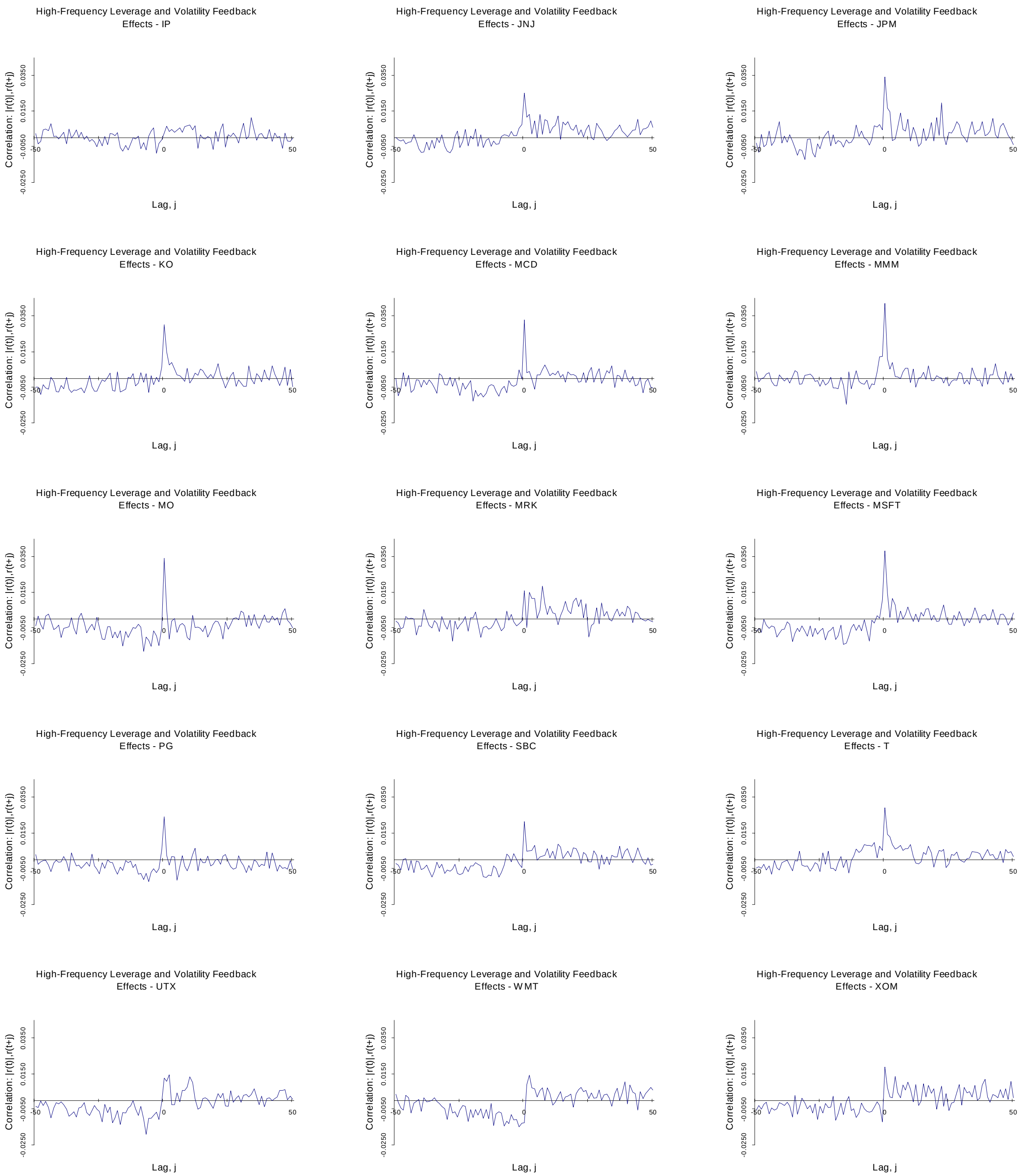


Figure 3 cont.: High-frequency leverage and volatility feedback effects, stocks IP-XOM



## Test for (Jump-Adjusted) Diffusion w/ or w/o Leverage

Results Motivate direct Test for whether Daily Returns Consistent w/ Underlying (Jump-Adjusted) Pure Diffusive Process = Continuous Local (Semi-)MG

Compute RV from HF Jump-Adjusted Returns, Fix $\tau^*$ [w/ Average Calendar Interval equal to (Multiple of) Trading Day], Sample in Financial Time

Then Test whether the RV-Standardized Financial Return Series is iid N(0,1) !
[ Peters and de Vilder (2004), henceforth PdV, did this w/o Jump-Adjustment ]

### Potential Complications

Realized (Jump-Adjusted) Volatility is Noisy Measure of Integrated Variance

RV measure moves Discretely w/ each non-zero Return so Overshooting of $\tau^*$

RV may move Fast, so Few Intraday Returns for Financial Return Calculation and the potential Measurement Errors Magnify

# Practical Testing Issues (from ABDobrev)

**General Properties of RV-Standardized Trading Day Returns**

**Thin-Tailed by Construction !!**

Let **n = 1/Δ** Intraday returns be available and **c** be the Trading Day Return

How Large can the RV-Standardized Return be?

Let $\mathbf{X} = \{(x_1, x_2, ..., x_n) : \sum_{j=1}^{n} x_j = c\}$ and Answer Solves

$$Max \sum_{j=1}^{n} r_{t+j\cdot\Delta,\Delta} \, / \, \sqrt{RV_{t+1}(\Delta)} = c \cdot (\sum_{j=1}^{n} r^2_{t+j\cdot\Delta,\Delta})^{-1/2}$$

subject to $(r_{t+\Delta,\Delta}, r_{t+2\Delta,\Delta}, ..., r_{t+(n-1)\Delta,\Delta}, r_{t+1,\Delta})$ belonging to the set X.

# Practical Testing Issues (continued)

The Solution is

$$r_{t+j\Delta,\Delta} = c/n, \qquad j = 1, 2, \ldots, n,$$

and the associated Maximum Attainable RV-Standardized Return is

$$c \cdot \left(\sum_{j=1}^{n} r^2_{t+j\cdot\Delta,\Delta}\right)^{-1/2} = \sqrt{n}$$

Note Solution independent of assumed **c** and only function of **n**.
Valid Universally - Jumps or not, Financial Time Returns or not.

All RV-Standardized Returns Fall within the $[-\sqrt{n}, \sqrt{n}]$ interval.

# Practical Testing Issues (continued)

RV-Standardized Returns are always Truncated with Thin Tails
However, for $n \to \infty$ Convergence to Standard Normal

## Do Jumps have an Impact?

If One "Dominant" Jump during Trading Day (same sign as Daily Return)
then Marginal Increase in Jump Size will Lower RV-Standardized Return

Intuitively, the Increase in Choppiness enhances the Denominator more than the Numerator and the Standardization Lowers the Normalized Return Observation

Hence, the **Conjecture** that **Jumps tend to Induce further Tail-Thinness** in the RV-Standardized Returns

Not always the case, as Jumps may be of Opposite Sign to Daily Return or different Returns may Offset each Other

# An Approximating Finite-Sample Distribution

Define RV-Standardized Returns by

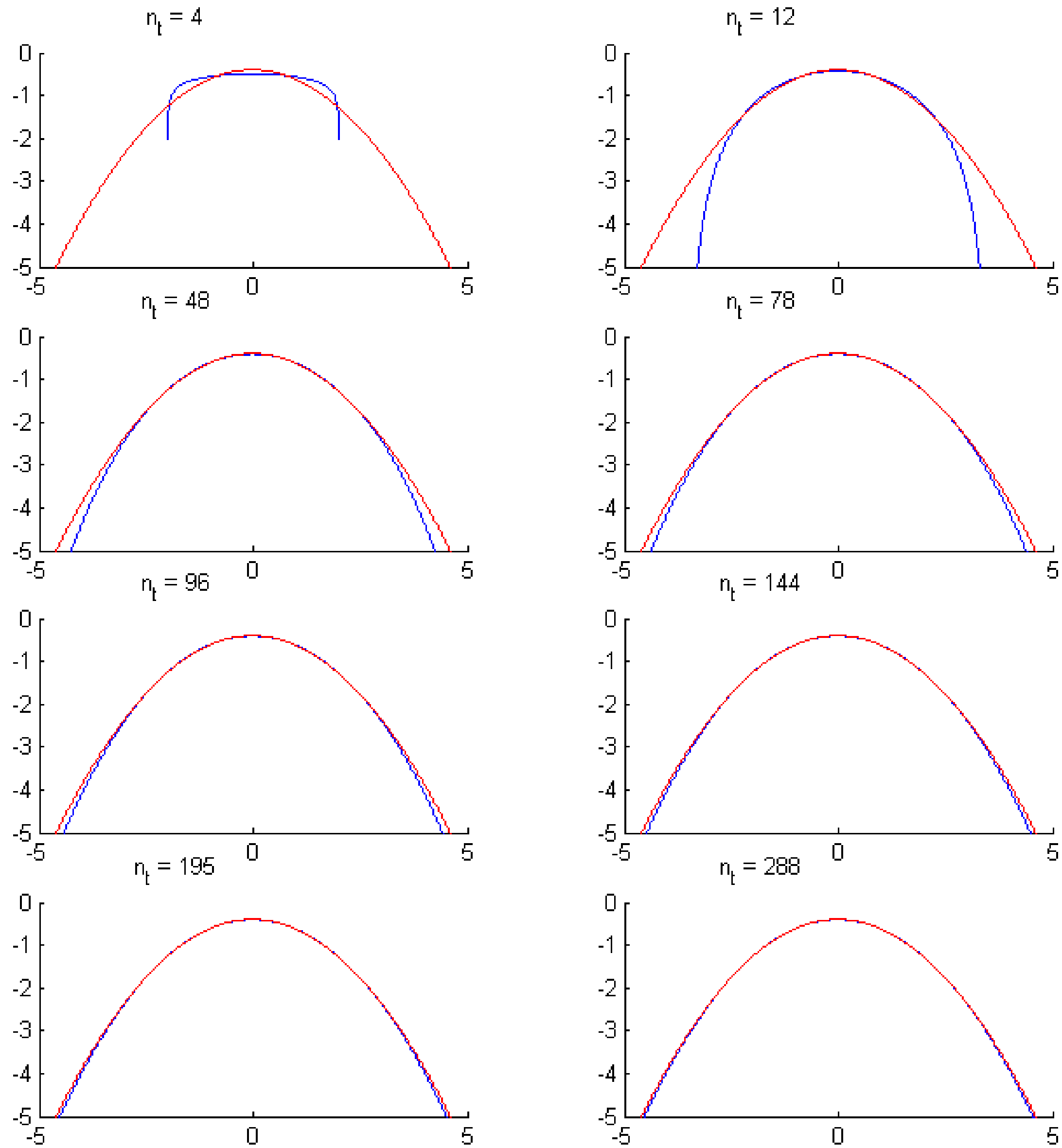
$$\tilde{R}_{t+1} \equiv \sum_{j=1}^{n} r_{t+j\cdot\Delta,\Delta} \Big/ \sqrt{RV_{t+1}(\Delta)}$$

Then in Pure Diffusive Setting (or for all Jumps perfectly Extracted) and
If Volatility is Constant over Trading Period then (PdV)

$$f_{\tilde{R}}(\tilde{r}) = \frac{\Gamma(n/2)}{\sqrt{\pi n}\ \Gamma((n-1)/2)} \left(1 - \frac{\tilde{r}^2}{n}\right)^{\frac{(n-3)}{2}} 1_{(-\sqrt{n},\sqrt{n})}(\tilde{r}),\ \tilde{r} \in \mathbb{R}$$

Density provides Finite-Sample Approximation as alternative to Asymptotic Standard Normal - likely better for Moderate-to-Low Values of **n**

May be Poor for Very Low values as Constant Volatility dubious


$n_t$ = 4
$n_t$ = 12
$n_t$ = 48
$n_t$ = 78
$n_t$ = 96
$n_t$ = 144
$n_t$ = 195
$n_t$ = 288

# Practical Testing Issues (continued)

One Popular Normality Test is the Jarque-Bera (JB) Test
Two Issues with JB Test in current setting

**Finite Sample Bias**, Approximating Distribution has Second Moment of One, but Fourth Moment of **$3n/(n+2)$**

Loss of Power as JB is generic Test for **$N(\mu, \sigma^2)$** Null, while Hypothesis of Interest here is a Sharp **iid $N(0,1)$** Hypothesis

JB Test procedure undertakes initial Demeaning and Rescaling

PdV use Fourth Moment of **$3n/(n+2)$** in JB Test to alleviate Finite-Sample Bias

We Implement **Joint Moment Tests** via **$1^{th}$-$4^{th}$ Moments of Standard Normal** and **Approximating Finite Sample Distribution** (w/o Centering and Rescaling)

ABDobrev (2005) show in Large-Scale Simulation latter is **quite Powerful Test** while the **usual JB Test is basically Useless** (and/or misleading)!!

# Distributional Evidence for DJIA Stock Returns

## Separate Evidence for Raw and Demeaned (Sample Mean) Trading Returns

1) Standardized by Unconditional Standard Deviation

2) Standardized by GARCH (1,1) Conditional Standard Deviation

3) Standardized by $RV^{1/2}$

4) Standardized by $CV^{1/2}$

5) Standardized by $CVS^{1/2}$

6) $R_k$ Standardized by $(\tau^*)^{1/2}$

7) $(R_k + ... + R_{k+4})$ Standardized by $(5 \cdot \tau^*)^{1/2}$

Table 5: Daily Return Distributions

| | Raw Returns Significance | | Demeaned Returns Significance | |
|---|---|---|---|---|
| Series | 5 % level | 1 % level | 5 % level | 1 % level |
| $R_t/\sqrt{Var(R_t)}$ | 30 | 30 | 30 | 30 |
| $R_t/\sqrt{GARCH(1,1)}$ | 30 | 30 | 30 | 30 |
| $R_t/\sqrt{RV_t}$ | 21 | 12 | 18 | 9 |
| $\tilde{R}_t/\sqrt{CV_t}$ | 18 | 10 | 15 | 9 |
| $\hat{R}_t/\sqrt{CVS_t}$ | 20 | 12 | 18 | 11 |
| $F\hat{R}_{1,t}/\sqrt{R\tau_1^*}$ | 13 | 6 | 11 | 5 |
| $F\hat{R}_{5,t}/\sqrt{R\tau_5^*}$ | 6 | 3 | 2 | 2 |

Note: The table reports the number of stocks (out of 30) in the period 1998-2002 for which the hypothesis of normality was rejected based on the combined test for all 4 moments. The results for the "Demeaned Returns" are based on demeaning the returns series in the numerator. $R_t$ is daily return, $\tilde{R}_t$ and $\hat{R}_t$ are daily returns with jumps removed (simple and sequential procedure), $RV_t$ is realized volatility, $CV_t$ and $CVS_t$ are continuous components of volatility based on the simple and sequential procedure, respectively. For $i = 1, 5$, $F\hat{R}_{i,t}$ is the financial returns based on the $\hat{R}_t$ series, and $R\tau_i^*$ is the realized $i \times \tau^*$. For details, see Table **??** of the appendix.

Table 6: Daily Return Distributions, Feb. 2001 - Dec. 2004

| | Raw Returns Significance | | Demeaned Returns Significance | |
|---|---|---|---|---|
| Series | 5 % level | 1 % level | 5 % level | 1 % level |
| $R_t/\sqrt{Var(R_t)}$ | 30 | 30 | 30 | 30 |
| $R_t/\sqrt{GARCH(1,1)}$ | 30 | 30 | 30 | 30 |
| $R_t/\sqrt{RV_t}$ | 15 | 10 | 12 | 9 |
| $\tilde{R}_t/\sqrt{CV_t}$ | 14 | 9 | 11 | 8 |
| $\hat{R}_t/\sqrt{CVS_t}$ | 13 | 10 | 12 | 9 |
| $F\hat{R}_{1,t}/\sqrt{\tau^*}$ | 8 | 6 | 7 | 5 |
| $F\hat{R}_{5,t}/\sqrt{5\tau^*}$ | 4 | 2 | 2 | 0 |

Note: The table reports the number of stocks (out of 30) in the period Feb. 2001 - Dec. 2004 for which the hypothesis of normality was rejected based on the combined test for all 4 moments. The results for the "Demeaned Returns" are based on demeaning the returns series in the numerator. $R_t$ is daily return, $\tilde{R}_t$ and $\hat{R}_t$ are daily returns with jumps removed (simple and sequential procedure), $RV_t$ is realized volatility, $CV_t$ and $CVS_t$ are continuous components of volatility based on the simple and sequential procedure, respectively. For $i = 1, 5$, $F\hat{R}_{i,t}$ is the financial returns based on the $\hat{R}_t$ series, and $R\tau_i^*$ is the realized $i \times \tau^*$.

Figure 5: QQ plots of daily returns for 30 DJIA stocks standardized by sample standard deviation

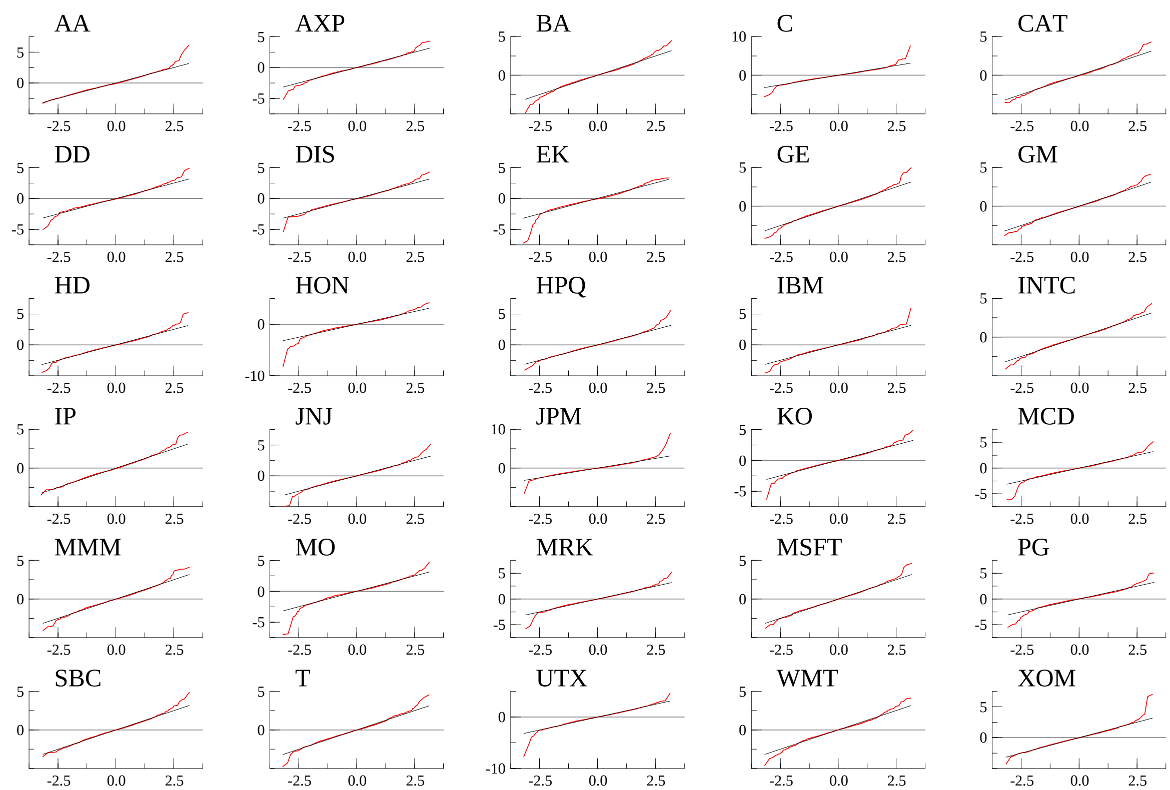


Figure 7: QQ plots of daily returns for 30 DJIA stocks standardized by GARCH(1,1) standard errors

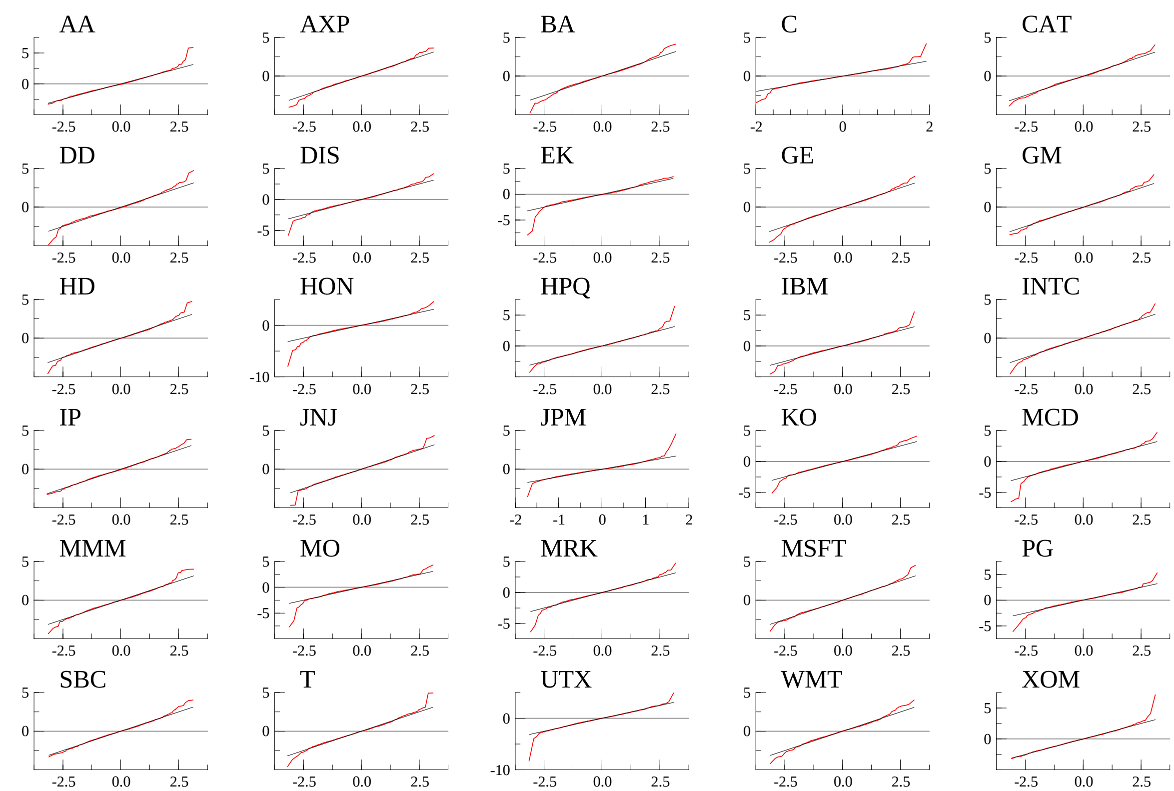


Figure 9: QQ plots of daily returns for 30 DJIA stocks standardized by realized volatility

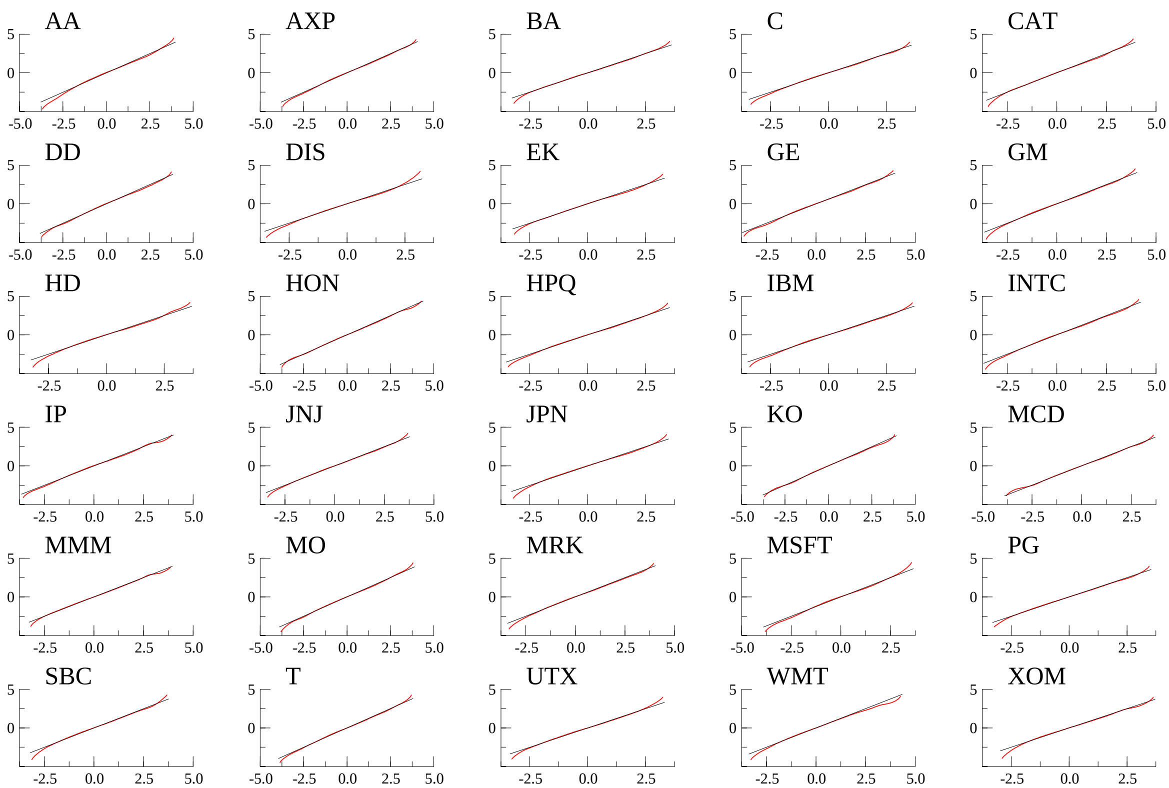


Figure 11: QQ plots of jump-adjusted (simple method) daily returns for 30 DJIA stocks standardized by continuous component of realized volatility

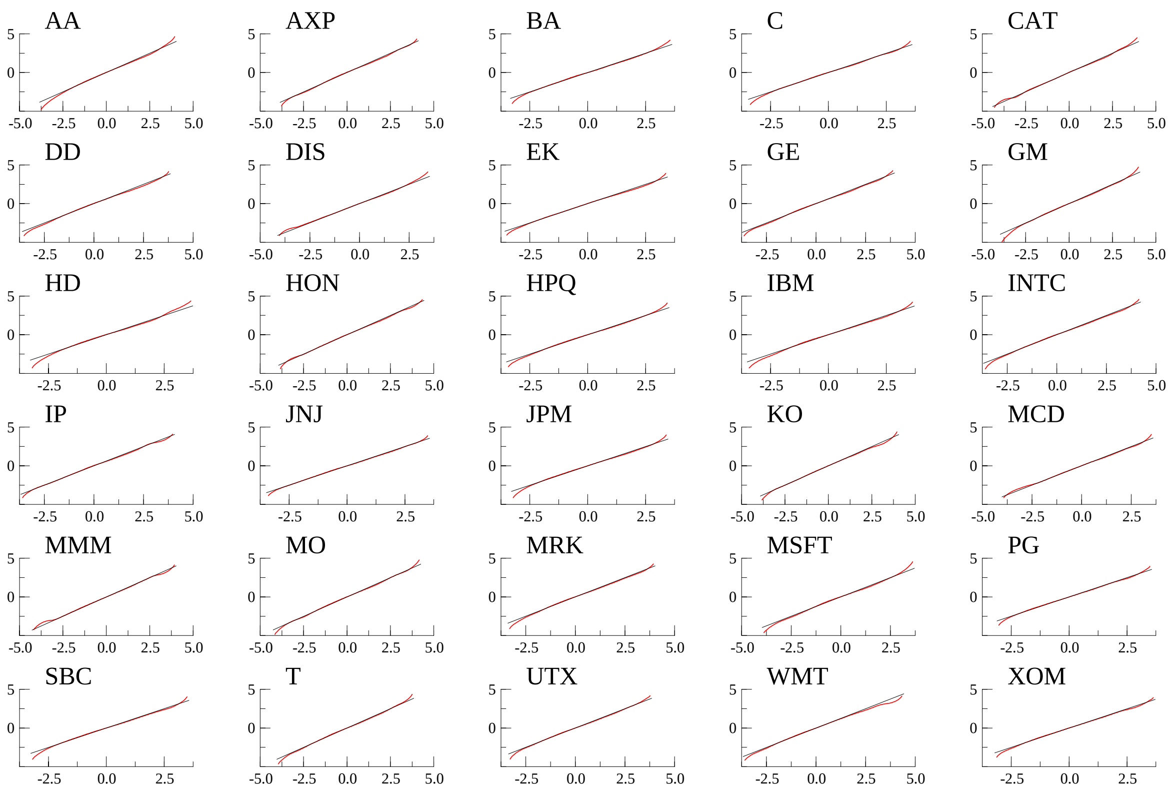


Figure 13: QQ plots of jump-adjusted (sequential method) daily returns for 30 DJIA stocks standardized by continuous component of realized volatility

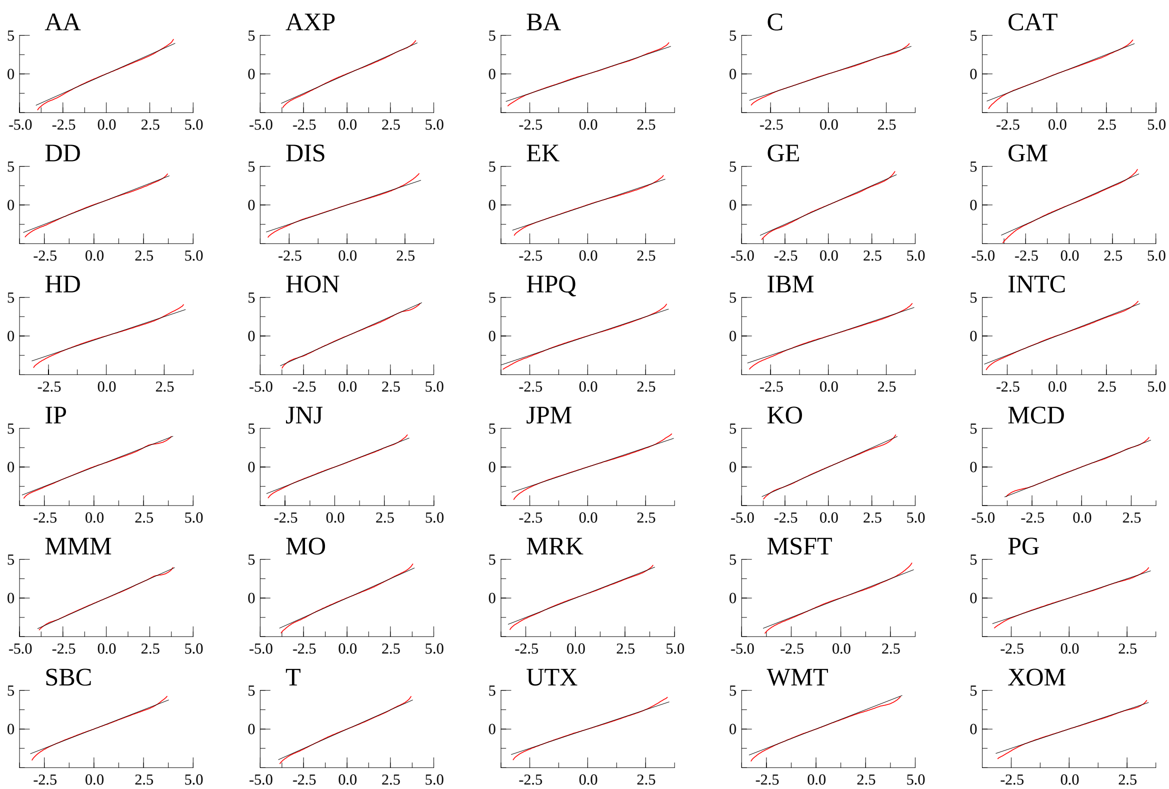


Figure 15: QQ plots of financial time daily returns for 30 DJIA stocks standardized by $\tau^*$

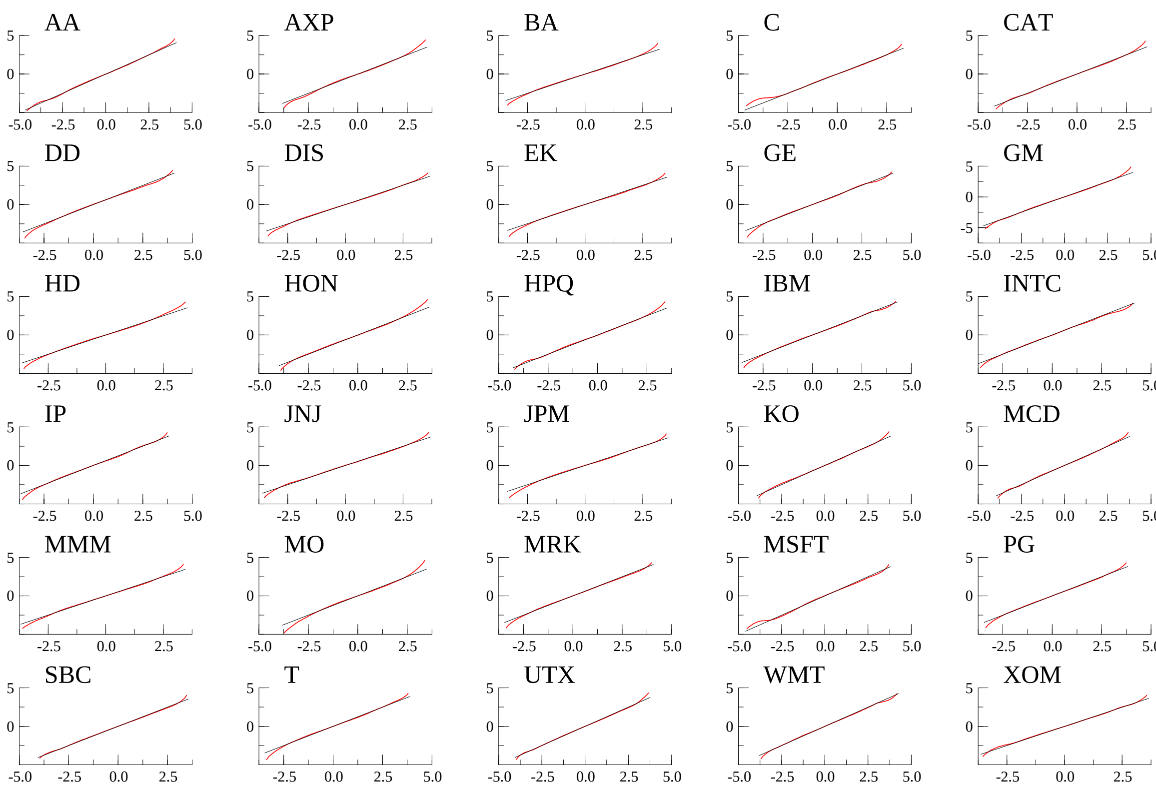


Figure 16: $p$-values for the 30 DJIA stocks, Jan. 1998 - Dec. 2002

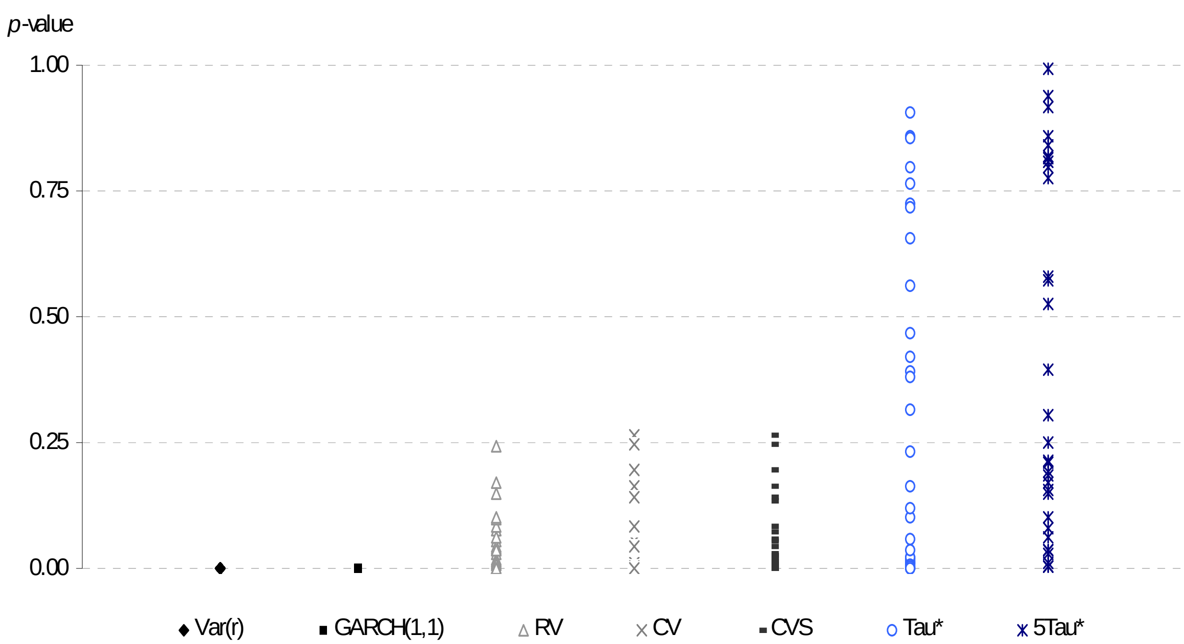


Figure 21: $p$-values for the 30 DJIA stocks, Feb. 2001 - Dec. 2004

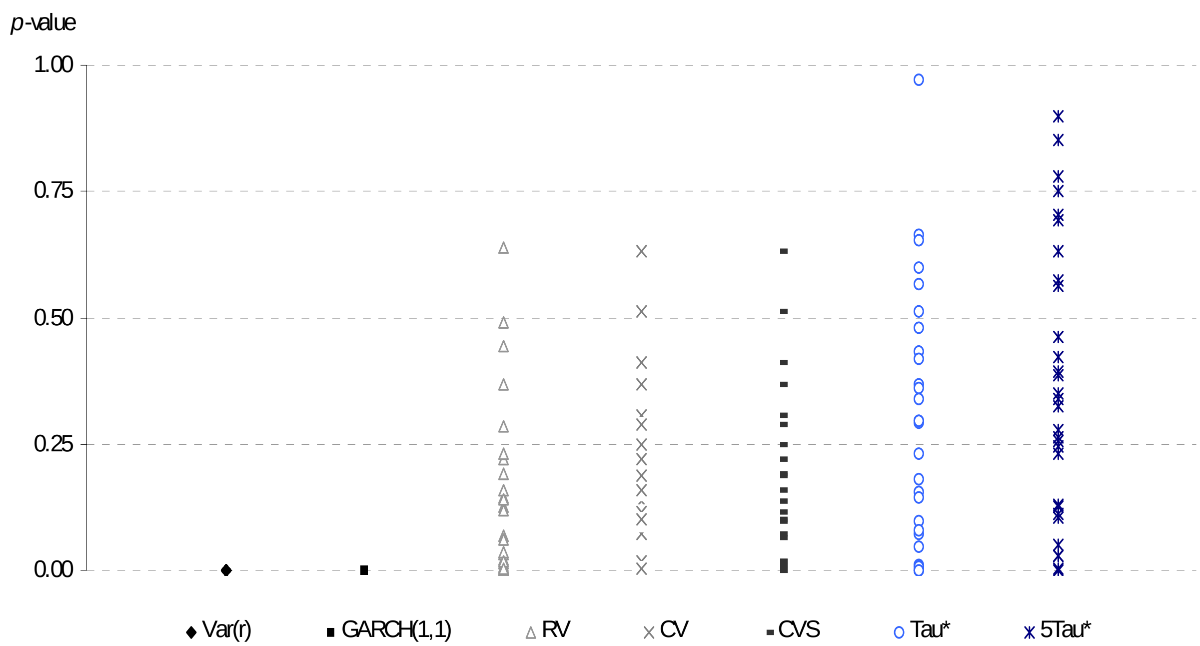
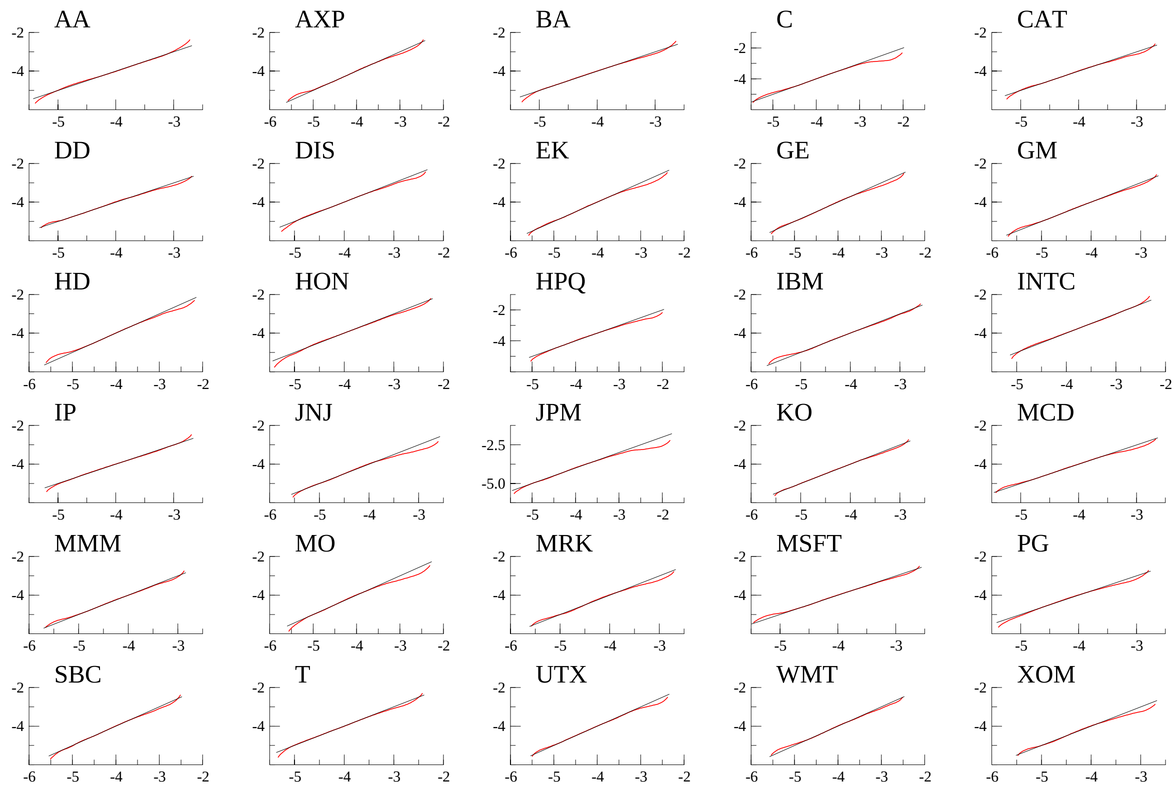


Table A7: Normality tests for volatility measures

| | $\ln RV_t$ | | $\ln CV_t$ | | $\ln CVS_t$ | |
|---|---|---|---|---|---|---|
| Ticker | JB | LV | JB | LV | JB | LV |
| AA | 0.3571 | 0.8155 | 0.0679 | 0.6884 | 0.1474 | 0.7444 |
| AXP | 0.0001** | 0.0423* | 0.0006** | 0.0985 | 0.0004** | 0.0968 |
| BA | 0.0091** | 0.2277 | 0.3606 | 0.7015 | 0.0896 | 0.4764 |
| C | 0.0000** | 0.0004** | 0.0000** | 0.0029** | 0.0000** | 0.0025** |
| CAT | 0.0003** | 0.0631 | 0.0041** | 0.0907 | 0.0018** | 0.0814 |
| DD | 0.0010* | 0.1247 | 0.0028** | 0.1705 | 0.0016** | 0.1491 |
| DIS | 0.0003** | 0.1696 | 0.4722 | 0.8322 | 0.1241 | 0.6436 |
| EK | 0.0000** | 0.0162* | 0.0000** | 0.0313* | 0.0000** | 0.0201* |
| GE | 0.0000** | 0.0251* | 0.0000** | 0.0251* | 0.0000** | 0.0185* |
| GM | 0.0105* | 0.3067 | 0.2150 | 0.6132 | 0.1256 | 0.5625 |
| HD | 0.0011** | 0.2371 | 0.0001** | 0.1087 | 0.0011** | 0.2022 |
| HON | 0.0108* | 0.5229 | 0.3198 | 0.8203 | 0.1082 | 0.7082 |
| HPQ | 0.0005** | 0.2415 | 0.0042** | 0.3654 | 0.0014** | 0.3020 |
| IBM | 0.0064** | 0.2328 | 0.4929 | 0.8283 | 0.4318 | 0.8047 |
| INTC | 0.4318 | 0.8937 | 0.5376 | 0.9415 | 0.5949 | 0.9433 |
| IP | 0.1295 | 0.5798 | 0.1792 | 0.5413 | 0.2343 | 0.6175 |
| JNJ | 0.0000** | 0.0000** | 0.0000** | 0.0000** | 0.0000** | 0.0001** |
| JPM | 0.0000** | 0.0065** | 0.0000** | 0.0201* | 0.0000** | 0.0143* |
| KO | 0.0535 | 0.6028 | 0.1467 | 0.6458 | 0.0992 | 0.6414 |
| MCD | 0.0000** | 0.0106* | 0.0004** | 0.0245* | 0.0000** | 0.0116* |
| MMM | 0.1825 | 0.6574 | 0.5995 | 0.8608 | 0.4942 | 0.8254 |
| MO | 0.0000** | 0.0000** | 0.0000** | 0.0000** | 0.0000** | 0.0004** |
| MRK | 0.0000** | 0.0000** | 0.0000** | 0.0001** | 0.0000** | 0.0000** |
| MSFT | 0.0282* | 0.3668 | 0.1609 | 0.5052 | 0.2241 | 0.6318 |
| PG | 0.0000** | 0.0110* | 0.0000** | 0.0110* | 0.0000** | 0.0429* |
| SBC | 0.0033** | 0.4038 | 0.0442* | 0.5856 | 0.0120* | 0.5019 |
| T | 0.2649 | 0.7385 | 0.4859 | 0.8071 | 0.3839 | 0.7780 |
| UTX | 0.0141* | 0.3199 | 0.4258 | 0.7748 | 0.3261 | 0.7398 |
| WMT | 0.0473 | 0.4915 | 0.0681 | 0.4732 | 0.0573 | 0.4755 |
| XOM | 0.0000** | 0.0006** | 0.0000** | 0.0008** | 0.0000** | 0.0011** |

Note: The table reports the $p$-values of test statistics for normality. $RV_t$ is realized volatility, $CV_t$ and $CVS_t$ are continuous components of volatility. JB is the Jarque & Bera (1987) test and LV is the generalized skewness-kurtosis test of Lobato & Velasco (2004). One and two asterisks denote rejection at 5% and 1% significance level, respectively.

Figure 20: QQ plots of log-continuous component (sequential method) of realized volatility for 30 DJIA stocks



## Some Interpretation of Distributional Results

**As Expected, Raw and GARCH Standardized Returns not Gaussian (1,2)**

**Significant Improvement from RV Standardization, but Overrejection (3)**

**Jump Adjustment has essentially further No Impact !! (4,5)**
- **There is a Sense in which Jumps Self-Standardize in RV Scaling**

**Financial Time Sampling has Large (positive) Impact**
- **Results are very Good (given presence of Noise), but not Perfect**

**Hence, Standard Normal Approximation strikingly Good in Financial Time**

**Evidence of Leverage is thus quite Striking**

**Evidence for Discontinuities overall overwhelming, but Implications for Distributional Properties more Limited (Theory and Simulation)**

**Diffusive Volatility is strikingly Close to Log-Normally Distributed**

## Concluding Observations

**Feasible to Gauge Critical Features of Daily Returns and Decompose into Jumps, Leverage Effect, Path of Diffusive Volatility from HF Return Data through Nonparametric Techniques under weak Auxiliary Assumptions**

**Each Step of Procedure speaks to Important Feature of Return Process**

**Serve as Informal Test for whether Return Process a Semi-Martingale**

**Extensive Simulation Evidence on what works and what not (ABDobrev)**

**Should be Useful in Longer Run for Modeling Daily Return Series**

**Here Only Claim to Shed Light on Relation b/w Real-Time Evolution in Realized Volatility Notions and Daily Return and Volatility Characteristics**

**Evidence Compatible w/ DJIA Stocks driven by Jump-Diffusion w/ Leverage**

## For Future Inquiry and Research

**How widely will the Approximate Standard Normal Property Apply?**
**- and Is the Financial Time Transformation working b/c of Leverage?**

**Each Step of the Procedure can be Extended or Improved**

**Improve RV Measurement in face of Market Microstructure Noise**

**Relate Decomposition (Integrated Variance, Jumps) to other MDH Activity Variables like Trading Volume, Transaction Frequency, Quotes, Spreads**

**Document Direct Usefulness for Financial Decision Making**