Simulation Methods for Lévy-Driven CARMA Stochastic Volatility Models

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1. Introduction

- Modeling a financial price series as forced by a stochastic volatility process has a long history.
- Recent strong nonparametric evidence for jumps.
- Barndorff-Nielsen and Shephard (2002) present non-Gaussian OU Models where volatility is driven by a pure-jump Lévy process with nonnegative increments.

- Brockwell (2001, ...) introduce a generalization to the Lévy-driven CARMA (continuous time autoregressive moving average) class of volatility models.

- These classes of models based on more general Lévy processes will supplant traditional Brownian-based processes in serious efforts to model the movements of financial price data at the <u>very</u> high frequency.

- Regardless of the estimation technique (Bayesian, SMM, EMM, Indirect Inference ...), it is clear that simulation will play a crucial role in the implementation of these newer classes of processes.

- We develop and assess practical schemes to simulate from Lévy-driven models for financial price dynamics.

- In the following, the volatility dynamics are governed by the Brockwell-style CARMA extension of the Barndorff-Nielsen and Shephard non-Gaussian OU setup.

- The returns process also contains a jump component, which is linked with the jump innovations in volatility in order to accommodate the so-called leverage effect and provide dependence between the jumps in the price and the volatility.

- We use multivariate simulation schemes based on series expansions that turn out to be considerably simpler to implement than schemes based on the tail mass of the Lévy measure.

- We introduce a two-dimensional mixture of gammas Lévy process that is extremely flexible and easy to use.

2. Lévy Processes

A Lévy process could be described as continuous time analogue to the random walk in discrete time. The formal definition of the process is: A stochastic process $\{L_t : t \geq 0\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, taking values in $\mathbb{R}^d$ is a Lèvy process if the following conditions are satisfied:

1. It has independent increments.

2. $L_0 = 0$ a.s.

3. The increments of the process are stationary.

4. It is stochastically continuous.

5. There is $\Omega_0 \in \mathcal{F}$ with $P(\Omega_0) = 1$, such that that for every $\omega \in \Omega_0$ $L_t(\omega)$ is $c\grave{a}dl\grave{a}g$ (i.e it is right continuous with left limits).

The Lévy processes span a huge class:

Brownian Motion $\longleftarrow\longrightarrow$ compound Poisson

3. Series Representations of Lévy Processes

Our work is based on the series representation of Lévy processes summarized in Rosiński (2001). If $L(t)$ is a pure jump Lévy process of bounded variation on $[0,1]$ with Lévy measure ν then it can be written as

$$L(t) = \sum_{i=1}^{\infty} H(\Gamma_i, V_i) 1_{(U_i \le t)}, \;\; t \in [0,1]$$

where

1. $H(\cdot,\cdot)$ is nonincreasing in its first argument,
2. $\Gamma_i = \Gamma_{i-1} + E_i$, $E_i \sim \text{Exp}(1)$,
3. V_i are iid with cdf $F(v)$,
4. U_i are iid Uniform$[0, 1]$.

$H(\cdot,\cdot)$ and $F(\cdot)$ are chosen such that

$$\nu(A) = \int_0^\infty \Pr(H(r,V) \in A)dr, \qquad \text{for} \quad \forall A \in \mathcal{B}(\mathbb{R}_0^d)$$

Note

1. For every Lévy process there could be more than one shot noise decomposition of the Lévy measure-choice determined by how easy is to simulate from F and how fast is to evaluate $H(\cdot, \cdot)$.

2. There is series representation for Lévy processes of infinite variation, but in general it requires centering of the sum.

Practical implementation of the series representation requires truncation of the series. Here we work with the approximations

$$L_\tau(t) = \sum_{\Gamma_i \le \tau} H(\Gamma_i, V_i) 1_{(U_i \le t)}$$

and keep in mind that $L_\tau(t)$ itself is a Lévy process such that

$$\lim_{\tau \to \infty} L_\tau(t) = L(t)$$

where the convergence is almost sure and uniform in t.

Everything works out for functionals of $L(t)$ of the form

$$X(t) = \int_0^t f(t-s)dL(s)$$

which are approximated (simulated) via the truncated expansion

$$X_\tau(t) = \sum_{\Gamma_i \le \tau} f(t-U_i)H(\Gamma_i, V_i)1_{(U_i \le t)}$$

Here is the computer code to generate the series expansion (also called the shot-noise representation) for the case where the kernel is

$$f(x) = e^{-\rho x}, \qquad \text{for} \quad x \geq 0$$

```
* Control variables c, lambda, tau, nbin, seed,
*   rho, and SIMMAX already initialized
*
*  Loop to generate shot noises
      gamlag = 0.0d0
      i=0
      do while ((gamlag .lt. tau) .and. (i .le. SIMMAX))
         i=i+1
         ur     = ran(seed)
         gam(i) = gamlag - DLOG(ur)
         gamlag = gam(i)
         u(i)   = ran(seed)
         bin(i) = INT( 1.0d0 + u(i)*nbin )
         H(i)   = (1.0/lambda)*DEXP(-gam(i)/c)
         nsim   = i
      enddo
```

```
* Check to see if we ran out of space before gam(i)>tau
      if (gam(nsim) .lt. tau) then
         write(*,'(a )') 'PROBLEM: SIMMAX too small
         stop
      endif
* *
* Initialize
      do j=1,nbin
         levy(j) = 0.0
         x(j)    = 0.0
         tbin(j) = DFLOAT(j)/DFLOAT(nbin)
      enddo
```

```
*  Main allocation loops to accumulate the shots
   do i=1,nsim
    do j=bin(i),nbin
     levy(j) = levy(j) + H(i)
     x(j)    = x(j) + DEXP(-rho*(tbin(j) - u(i)) )*H(i)
    enddo
   enddo
```

4. Lévy Driven CARMA Models

The model for the financial price $p(t)$ is

$$dp(t) = \mu dt + \sigma(t-)dW(t) + dL_p(t)$$

$$a(D)\sigma^2(t) = b(D)DL_\sigma(t)$$

where $W(t)$ is a standard Brownian Motion, $L_p(t)$ is the pure jump Lévy process in the price, $L_\sigma(t)$ is the Lévy process for the stochastic volatility, D is a differential operator, and

$a(D)$ and $b(D)$ are given by

$$\begin{aligned} a(z) &= z^p + a_1 z^{p-1} + ... + a_p \\ b(z) &= b_0 + b_1 z + ... + b_q z^q, \quad q < p. \end{aligned}$$

Here $\sigma^2(t)$ is a Lévy-driven CARMA(p, q) process of Brockwell (2001). The solution is

$$\sigma^2(t) = \int_0^t g(t-u) dL_\sigma(u)$$

We focus attention on a CARMA$(2,1)$ process for the stochastic volatility, which we parameterize as

$$a(z) = (z - \rho_1)(z - \rho_2), \quad b(z) = 1 + b_1 z,$$

for real and distinct $\rho_1 < 0$ and $\rho_2 < 0$.

The kernel is

$$g(h) = \frac{1 + b_1\rho_1}{\rho_1 - \rho_2} e^{\rho_1 h} + \frac{1 + b_1\rho_2}{\rho_2 - \rho_1} e^{\rho_2 h}, \qquad h \geq 0$$

A necessary and sufficient condition guaranteing nonnegativity of the kernel in this case is $0 \leq b_1 \leq \max\{-1/\rho_1, -1/\rho_2\}$.

5. Simulations from Lévy driven CARMA Models

The stochastic volatility model implies that the return, $r_a(t)$, over the interval $(t-a, t]$ is

$$r_a(t) = p(t) - p(t-a) = \int_{t-a}^{t} \sigma(s-)\, dW(s) + L_p(t) - L_p(t-a$$

Since $\int_{t-a}^{t} \sigma(s-)\, dW(s)$ is Gaussian conditional on the pure jump processes in the price and the variance, we can write

$$p(t)-p(t-a) \stackrel{d}{=} Z_t\sqrt{\int_{t-a}^{t}\sigma^2(s)ds}+L_p(t)-L_p(t-a)$$

where $Z(t)$ is standard normal variable independent of $L_p(t)$ and $L_\sigma(t)$ and $\int_{t-a}^{t}\sigma^2(s)ds$ is the integrated variance over the time interval a.

By the Fubini theorem the integrated variance is

$$\int_{t-a}^{t} \sigma^2(s)ds = \int_{t-a}^{t} \int_{0}^{s} g(s-u)dL(u)ds$$

$$= \int_{t-a}^{t} \int_{u}^{t} g(s-u)dsdL(u) + \int_{0}^{t-a} \int_{t-a}^{t} g(s-u)dsdL(u)$$

$$= \int_{0}^{t} g^*(t,u)dL(u)$$

where the functional form of g^* can be obtained from that of g.

In the case of CARMA(2,1), the expression for the integrated variance is

$$\int_{t-a}^{t} \sigma^2(s)ds = \int_0^t g^*(t,u)dL(u)$$

where $g^*(t,u)$ is

$$\begin{cases} (e^{\rho_1(t-u)} - e^{\rho_1(t-u-a)})\frac{1+b_1\rho_1}{\rho_1(\rho_1-\rho_2)} + (e^{\rho_2(t-u)} - e^{\rho_2(t-u-a)})\frac{1+b_1\rho_2}{\rho_2(\rho_2-\rho_1)} \\ \qquad\qquad\qquad \text{if } 0 < u < t-a \\ (e^{\rho_1(t-u)} - 1)\frac{1+b_1\rho_1}{\rho_1(\rho_1-\rho_2)} + (e^{\rho_2(t-u)} - 1)\frac{1+b_1\rho_2}{\rho_2(\rho_2-\rho_1)} \\ \qquad\qquad\qquad \text{if } t-a \le u \le t \end{cases}$$

NOTE: We can simulate directly from the integrated variance process, a generalization of an observation by Barndorff-Nielsen and Shephard for the non-Gaussian OU model.

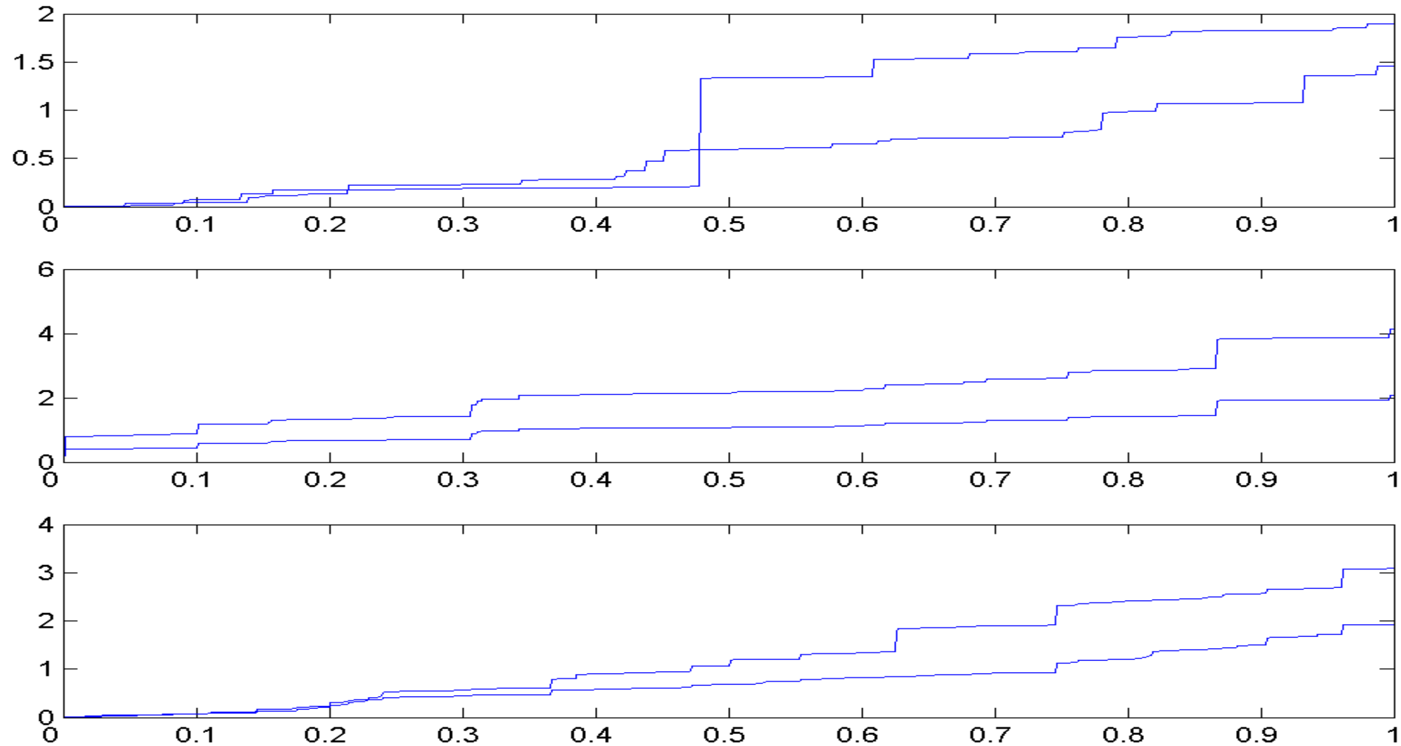
The generic simulation scheme for the price process at intervals of length a is:

1. Simulate jointly from the two pure jump Lévy processes $L_p(t)$ and $L_\sigma(t)$
2. Generate the implied integrated variance
3. Simulate the price increment by drawing from a normal distribution and add in the increments of $L_p(t)$ that occur within the interval $(t-a, t]$.

Some Examples

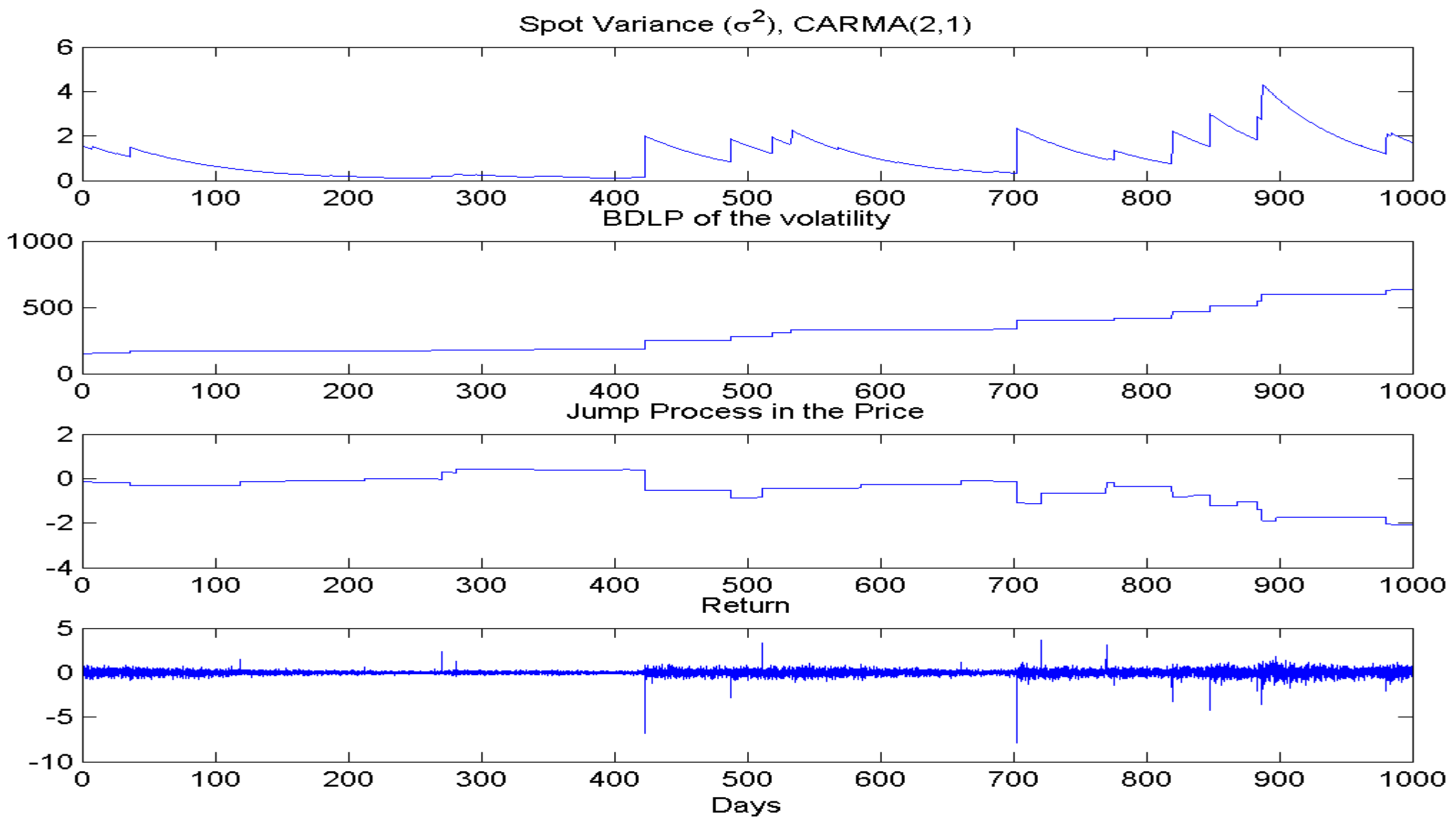
Different degrees of dependence between the two parts of a two dimensional mixture of gammas Lévy process.

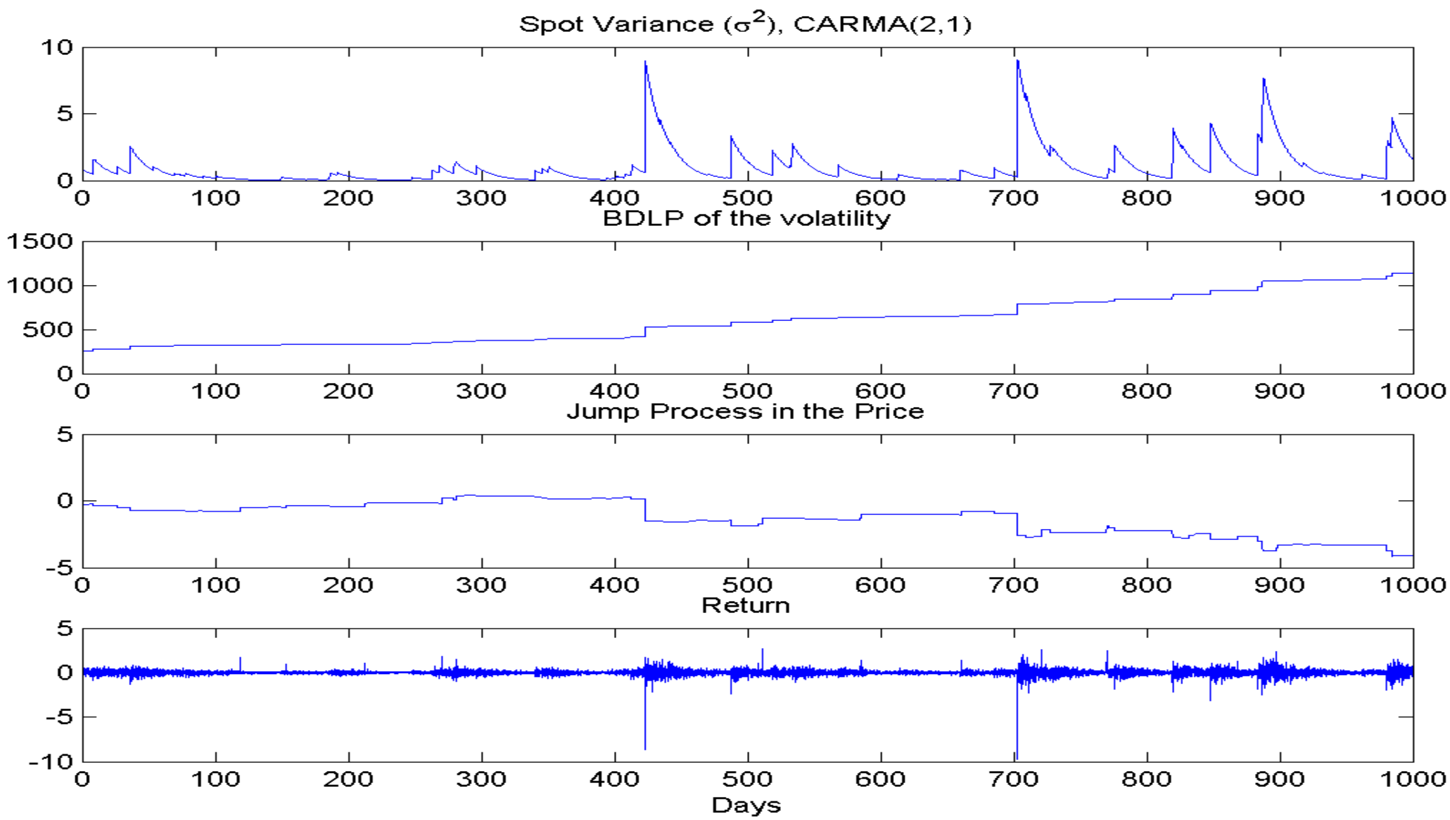
The top panel shows the complete independence case; the second one illustrates the complete dependence case; the third panel shows the general case.

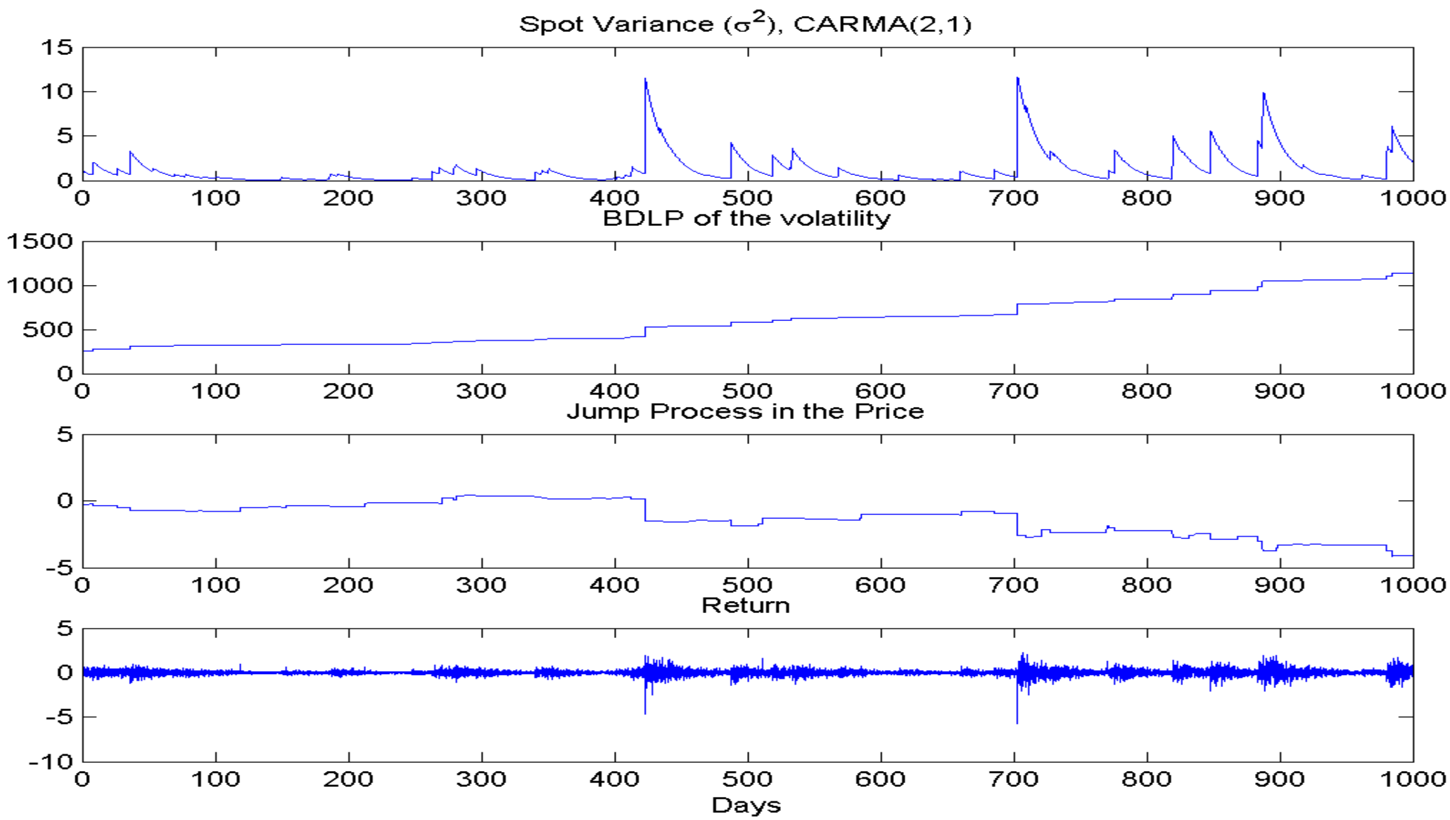

2
1.5
1
0.5
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
6
4
2
0
4
3
2
1
0

Simulated Half Hour Realizations from mixture of gammas driven CARMA(2,1) stochastic volatility model, various parameterizations.

The top panel shows the spot volatility; the second illustrates the Lévy subordinator driving the volatility; the third shows the pure jump part of the price process and the bottom panel shows the half hour price change.

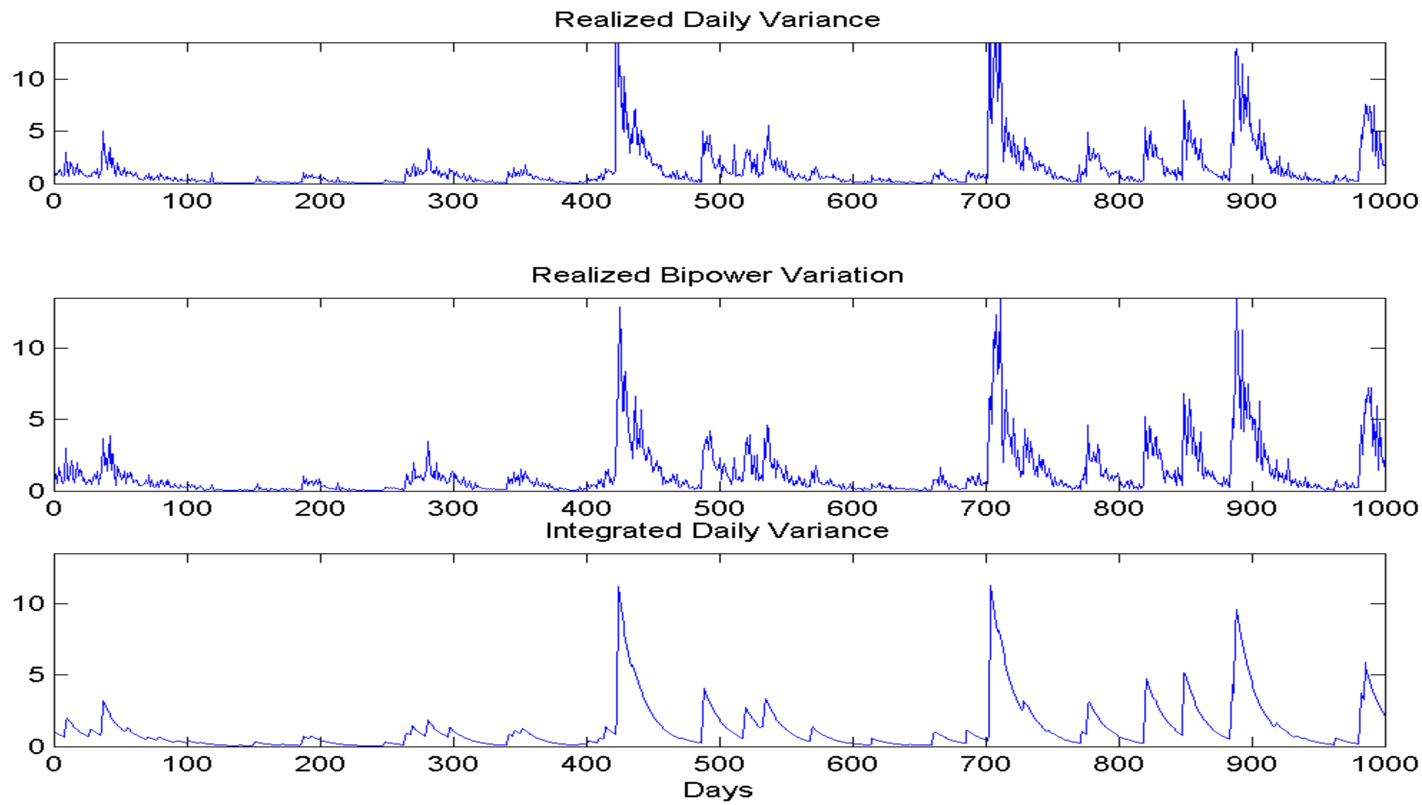

Spot Variance (σ2), CARMA(2,1)
6
4
2
0
0 100 200 300 400 500 600 700 800 900 1000
BDLP of the volatility
1000
500
0
0 100 200 300 400 500 600 700 800 900 1000
Jump Process in the Price
2
0
-2
-4
0 100 200 300 400 500 600 700 800 900 1000
Return
5
0
-5
-10
0 100 200 300 400 500 600 700 800 900 1000
Days


Spot Variance (σ²), CARMA(2,1)
10
5
0
0 100 200 300 400 500 600 700 800 900 1000
BDLP of the volatility
1500
1000
500
0
0 100 200 300 400 500 600 700 800 900 1000
Jump Process in the Price
5
0
-5
0 100 200 300 400 500 600 700 800 900 1000
Return
5
0
-5
-10
0 100 200 300 400 500 600 700 800 900 1000
Days


Spot Variance (σ^2), CARMA(2,1)
15
10
5
0
0 100 200 300 400 500 600 700 800 900 1000
BDLP of the volatility
1500
1000
500
0
0 100 200 300 400 500 600 700 800 900 1000
Jump Process in the Price
5
0
-5
0 100 200 300 400 500 600 700 800 900 1000
Return
5
0
-5
-10
0 100 200 300 400 500 600 700 800 900 1000
Days

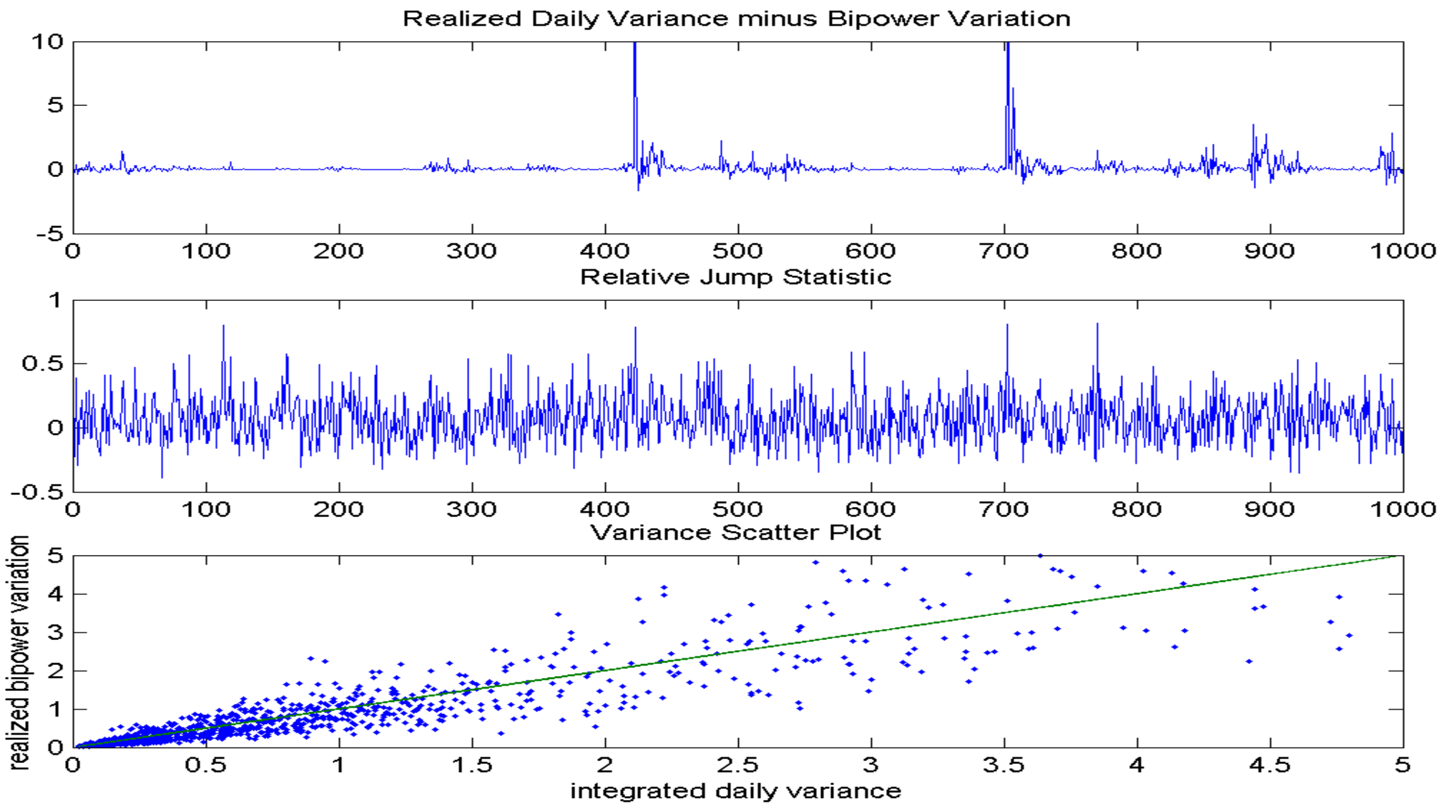
Summary statistics for the simulated mixture of gammas CARMA(2,1) stochastic volatility model.

The top panel shows the realized daily variance, the middle shows the bipower variation and the third panel shows the integrated daily variance.


Realized Daily Variance
10
5
0
0
100
200
300
400
500
600
700
800
900
1000
Realized Bipower Variation
10
5
0
0
100
200
300
400
500
600
700
800
900
1000
Integrated Daily Variance
10
5
0
0
100
200
300
400
500
600
700
800
900
1000
Days

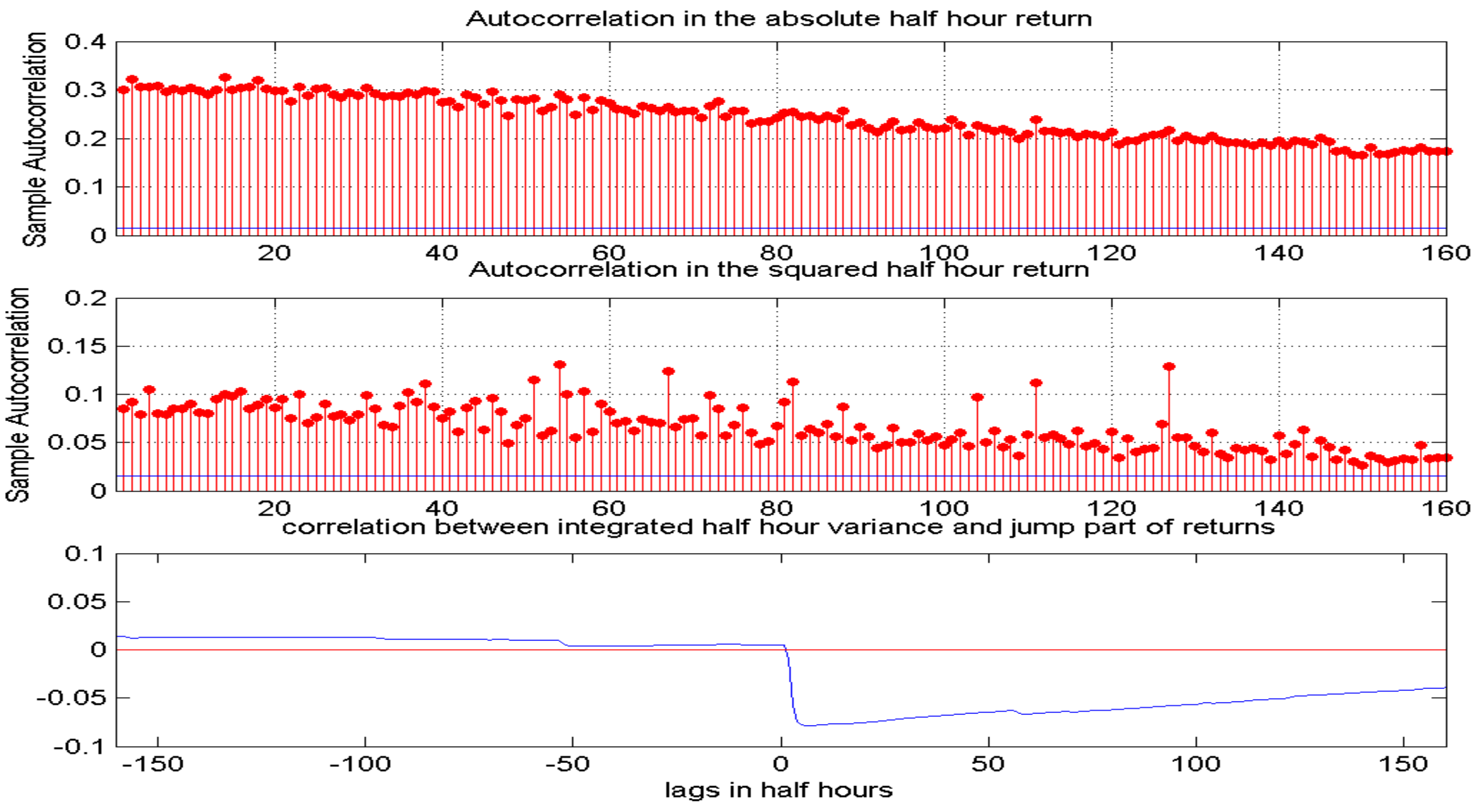
Volatility summary statistics for the simulated mixture of gammas CARMA(2,1) stochastic volatility model.

The top panel shows the difference between the realized volatility and the bipower variation; the middle is a plot of the relative jump statistic; the bottom panel plots the bipower variation against the integrated daily variance.


Realized Daily Variance minus Bipower Variation
10
5
0
-5
0 100 200 300 400 500 600 700 800 900 1000
Relative Jump Statistic
1
0.5
0
-0.5
0 100 200 300 400 500 600 700 800 900 1000
Variance Scatter Plot
realized bipower variation
5
4
3
2
1
0
0 0.5 1 1.5 2 2.5 3 3.5 4 4.5 5
integrated daily variance

Dependence measures for the simulated mixture of gammas CARMA(2,1) stochastic volatility model.

The top panel shows the autocorrelation in the absolute half hour returns; the second panel shows the autocorrelation in the squared half hour returns; the third shows the serial cross correlation between the increment in the price jump process and lags and leads of the integrated volatility


Autocorrelation in the absolute half hour return
Sample Autocorrelation
0.4
0.3
0.2
0.1
0
20 40 60 80 100 120 140 160
Autocorrelation in the squared half hour return
Sample Autocorrelation
0.2
0.15
0.1
0.05
0
20 40 60 80 100 120 140 160
correlation between integrated half hour variance and jump part of returns
0.1
0.05
0
-0.05
-0.1
-150 -100 -50 0 50 100 150
lags in half hours

6. Generalization to Lévy-Driven Continuous Time Models

The simulation schemes easily generalize to Lévy-Driven SDE-s

$$dX(t) = \alpha(X(t))dt + f(X(t-))dL(t)$$

The Euler discretization is

$$\begin{aligned} X(t) &= X(t-\delta) + \alpha[X(t-\delta)]\delta + \\ & \quad f[X(t-\delta)] \underbrace{(L(t) - L(t-\delta))} \end{aligned}$$

Simulate the increment

$$L(t) - L(t-\delta)$$

using the series approximation described above.

6. Conclusions

- Schemes based on series expansions are very efficient and easy to program.
- In the multivariate context the schemes are much simpler and easier to implement than copulas.
- Empirical work remains a challenge: Bayes, SMM, Indirect Inference, EMM, ... all entail simulation in some way or another.