Econometric Analysis of Jump-Driven Stochastic Volatility Models

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Abstract

This paper defines and develops a general new class of jump-driven stochastic volatility models. I focus attention on two particular semiparametric classes of jump-driven stochastic volatility models. In the first the price has a continuous component with time varying volatility and time homogenous jumps. The second jump-driven stochastic volatility model analyzed here has only jumps in the price, which exhibit time variation. I derive moments related with the two models. In the empirical application I model the memory of the stochastic variance with a CARMA(2,1) kernel and estimate the models by matching power variation statistics calculated from high frequency data. The models containing continuous component in the price are found to perform better for the particular jump specification.

Keywords: Lévy process, method of moments, power variation, stochastic volatility, realized variance, quadratic variation.

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1 Introduction

Continuous time stochastic volatility models have been used for a long time in the theoretical and empirical finance literature. A very general stochastic volatility model for the logarithmic price of a financial asset $p(t)$, which incorporates almost all existing stochastic volatility models, could be written as

$$p(t) = p(0) + \int_0^t \alpha(s)ds + \int_0^t \sigma_1(s-)dW(s) + \int_0^t \int_{\mathbb{R}_0^n} \sigma_2(s-)g(x)\tilde{\mu}(ds, dx), \tag{1}$$

where $W(t)$ is a standard Brownian motion, μ is a integer valued random measure on $\mathbb{R} \times \mathbb{R}_0^n$ ($\mathbb{R}_0^n := \mathbb{R}^n \backslash \{0\}$) and $\tilde{\mu}$ is its compensated version (see Ikeda and Watanabe (1981) and Jacod and Shiryaev (2003) for definitions and properties of integrals with respect to compensated integer valued random measures); $g : \mathbb{R}_0^n \to \mathbb{R}_0$ and $\sigma_1(t)$ and $\sigma_2(t)$ are two one dimensional processes such that the integrals above are well defined. The first component of the price is the drift, which could be potentially stochastic. The second component is a continuous martingale, while the third one is a discontinuous martingale. The integer valued measure μ controls the jumps in the price process. Most of the stochastic volatility models in the financial literature have the jumps in the price being of finite activity (for example compound Poisson). Defining the last term in (1) as an integral with respect to the compensated measure $\tilde{\mu}$ allows for considering much more general cases where the jump process in the price could be even of infinite variation. The state variables $\sigma_1(t)$ and $\sigma_2(t)$ are two stochastic processes which determine the time variation in the continuous and discontinuous martingale components of the price. [In the case of the discontinuous martingale time variation could be generated also if the measure μ is not time homogenous.] These state variables could be potentially linked with each other.

The main focus of this paper is stochastic volatility models in which the state variables when time varying are moving averages of positive jumps. The first type of jump-driven stochastic volatility (hereafter JDSV) models analyzed here is one in which the price comprises of a continuous component and jumps which are time homogenous. Models in this class were first introduced in Barndorff-Nielsen and Shephard (2001, 2002). Other models in this class are the extensions of the Barndorff-Nielsen and Shephard (2001, 2002) model proposed in Brockwell (2001a,b) and Todorov and Tauchen (2005). In general in these models, the continuous component plays a leading role. In contrast in the second class of JDSV models, I analyze here, the price is solely driven by jumps. The jumps however could (and in general do) exhibit time variation in these models. The COGARCH model of Klüppelberg, Lindner, and Maller (2004) without continuous component in the price exhibit qualitatively very similar behavior but are not nested in this class.

I estimate these models using high frequency data from the financial markets. The high frequency data should be naturally preferred from a statistical point of view. It provides more information and thus can yield much more efficient estimates as compared with daily data. Moreover the first type of JDSV models analyzed here has a continuous and a jump component in it. Therefore in order to separate them apart we should use high frequency data, as the behavior of the two components is different at the high frequencies. However, the presence of market microstructure noise (due to different factors such as irregular trading, discreteness of prices etc, see O'Hara (1995) and Hasbrouck (2004)) limits the usefulness of such data. The effect of the market microstructure noise is strongest at the highest frequencies, where it could completely dominate the fundamental price - see Bandi and Russell (2005), Hansen and Lunde (2005, 2006), Zhang et al. (2005), Barndorff-Nielsen et al. (2004) (these studies analyze the impact of the market microstructure noise on the estimate of the quadratic variation). There are two alternative ways to take into account the effect of the market microstructure noise. The first approach is to model it directly. Such modelling will allow to use the highest frequency and will also give estimates for the market microstructure noise. An alternative way is to work at relatively high frequencies at which however the effect of the market microstructure is negligible. At these frequencies the modelling of the market microstructure noise will be unnecessary. Here I follow the second approach and I do not model the market microstructure. I use the high frequency data to compute aggregated on daily level power variation statistics. The power variation statistics of a continuous and a discontinuous process have different properties and as a result, these statistics could be used to separate the jumps from continuous movements in the price.

Estimation of the JDSV models poses serious challenges. The presence of latent state variables makes direct maximum likelihood methods difficult to implement. Moreover in the very general form of the models I consider here, it might be not possible to put the price in a finite state Markov process. Therefore methods based on treating the unobserved states in the Markov process as parameters (such as those used in Eraker, Johannes, and Polson (2003), Roberts, Papaspiliopoulos, and Dellaportas (2004)) will not be applicable here in general. Additional difficulty with stochastic volatility models with jumps is that in only a few cases is the likelihood of the jump processes available in closed form. On the other hand when the integer valued random measure μ determining the jumps is homogenous Poisson, the JDSV models have a certain analytical tractability. Using it I can derive moments of the return process, which are important in identifying in a relatively efficient way the components of the model. This makes possible estimation based on these moments. The estimation approach applied here is similar to the one used in Bollerslev and Zhou (2002) (where they use the unobserved integrated variance) and Jiang and Oomen (2004) (where they use unbiased estimators for the variance) in estimating affine jump diffusion models using high frequency data.

In the empirical part of the paper I model the memory of the realized variance using a CARMA(2,1) kernel, which intuitively is a continuous time analogue of the ARMA(2,1) process in discrete time. For the jump specification I set the jumps in the variance to be proportional to the squared jumps in the price. In this setting I compare the performance of the JDSV models with the traditional affine jump diffusion models in which the state variables are following a square root process.

The rest of the paper is organized as follows. Section 2 analyzes the JDSV model where the price contains a continuous component, exhibiting time variation, and time homogenous jumps (called jump-diffusion JDSV model). Section 3 introduces and analyzes the pure jump JDSV model. Section 4 estimates the JDSV models as well as affine jump-diffusion models and compares their performance. In Subsection 4.1 I introduce the power variation statistics and provide details on the estimation method. Subsection 4.2 introduces the CARMA(2,1) kernel, which is used to model the memory of the stochastic variance, and Subsections 4.4.1 and 4.4.2 provide details on the jump specification in the jump-diffusion and pure jump JDSV models respectively. Subsection 4.4.3 introduces the affine jump-diffusion model and provides details on its estimation. In Subsection 4.4.4 I discuss the estimation results. Section 5 concludes the paper. The proofs of all theorems in the paper are relegated to Appendixes at the end of the paper.

2 Jump-Diffusion JDSV Model

The jump-diffusion JDSV model could be written in its general form as

$$p(t) = p(0) + \alpha t + \int_0^t \sigma(s-)dW(s) + \int_0^t \int_{\mathbb{R}_0^n} g(\mathbf{x})\tilde{\mu}(ds, d\mathbf{x}), \tag{2}$$

$$\sigma^2(t) = \int_{-\infty}^t \int_{\mathbb{R}_0^n} f(t-s)k(\mathbf{x})\mu(ds, d\mathbf{x}), \tag{3}$$

where $p(t)$ denotes the logarithm of the price of a financial asset at time t. I assume that $f(\cdot)$ and $k(\mathbf{x})$ take only positive values, which guarantees that the integral in (3) is always positive. In equations (2)-(3) μ denotes homogenous Poisson random measure on $\mathbb{R} \times \mathbb{R}_0^n$, its compensator is $\nu(ds, d\mathbf{x}) = dsG(d\mathbf{x})$ and $\tilde{\mu}$ is the compensated version of μ, that is $\tilde{\mu} = \mu - \nu$.

The financial asset prices in this model have two components - one is continuous and the other one is discontinuous and thus justifying the name of the model. The discontinuous part could be of finite variation, but could be also of infinite variation. This depends on the choice of the Poisson random measure and the function $g(\mathbf{x})$, since the jump part in the price equation is modelled as an integral with respect to $\tilde{\mu}$ (and not μ).

The stochastic variance $\sigma^2(t)$ in (3) is represented as an infinite moving average of positive valued jumps. The memory function (kernel) $f(\cdot)$ determines the effect of the past and current jumps on the current level of the stochastic variance. The choice of the kernel $f(\cdot)$ will determine the pattern of the autocorrelation in the return's variance. Drawing an analogy with time series modelling in discrete time a natural choice for $f(\cdot)$ would be a CARMA (continuous time autoregressive moving average) kernel analyzed in Brockwell (2001a,b), which intuitively is a continuous time analogue of the ARMA process in discrete time. However kernels which generate power decaying autocorrelation patterns are also an attractive alternative given the long memory type behavior of the return's variance. Examples of such kernels were discussed in Brockwell and Marquardt (2005).

In addition the representation of the stochastic variance as an integral with respect to positive valued jumps has the natural ability of generating sudden big moves in the variance. This was found empirically important in the studies of Chernov, Gallant, Ghysels, and Tauchen (2003) and Eraker, Johannes, and Polson (2003) among others. Therefore, from an empirical point of view the moving average specification of the stochastic variance in (3) seems pretty flexible and reasonable. On the other hand this modeling of the stochastic variance is particularly attractive as it avoids nonlinear transformations of underlying stochastic processes (like in the standard log-normal stochastic volatility model for example). This makes possible deriving moving average type representation of the integrated variance (introduced below), which is a key for many of the subsequent results.

Jumps play a leading role in the stochastic volatility model given in (2)-(3). The jumps in the price and in the variance of the price process are respectively

$$\Delta p(t) = g(\mathbf{x}), \tag{4}$$

$$\Delta\sigma^2(t) = f(0)k(\mathbf{x}). \tag{5}$$

The model could capture the so called leverage effect, observed in the stock markets, by linking the jumps in the price process and the jumps in the stochastic variance. This could be done for an appropriate choice of the functions $g(\mathbf{x})$ and $k(\mathbf{x})$. Also different choices for these functions could give different functional form of the relationship between the jumps in the price and in the variance. Such a way of introducing leverage effect through jumps is similar to the one used in Barndorff-Nielsen and Shephard (2001) and Carr, Geman, Madan, and Yor (2003), where jumps in price and variance are perfectly dependent. However here I allow for much more flexible dependencies between the jumps, which includes also the cases analyzed in Todorov and Tauchen (2005) (see also the discussion in Carr and Wu (2004) for modelling leverage effect through jumps).

From equation (4) we can see one possibly limiting feature of the stochastic volatility model given in (2)-(3). Mainly the jumps do not exhibit time variation. This is in contrast with the diffusive component of the price process, whose time variation is determined by the state variable $\sigma(t)$. In the second class of JDSV models I analyze in this paper I will allow for the jumps in the price to have time variation.

I proceed with stating the conditions guaranteeing the existence of the stochastic integrals used in defining the logarithm of the price process in the jump-diffusion JDSV model. The condition for the stochastic integral with respect to the Brownian motion to exist in $L_2(\Omega)$ sense is

$$\mathbb{E}\left(\int_0^t \sigma^2(s-)ds\right) < \infty, \qquad \text{for } \forall t < \infty. \tag{6}$$

By analogy for the stochastic integral with respect to the martingale integer valued random measure $\tilde{\mu}$ to exist in $L_2(\Omega)$ sense we need

H1. $\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}) < \infty$.

This condition is weaker than the one requiring the existence of the integral separately with respect to μ (in $L_2(\Omega)$ sense).

The integral in the stochastic variance is with respect to the measure μ itself (and not with respect to its compensated version) and for existence and weak stationarity of the process $\sigma^2(s)$ I assume the following is true

H2. $\int_0^\infty f(s)ds < \infty$ and $\int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x}) < \infty$,

H3. $\int_0^\infty f^2(s)ds < \infty$ and $\int_{\mathbb{R}_0^n} k^2(\mathbf{x})G(d\mathbf{x}) < \infty$.

When conditions **H2** and **H3** are satisfied, then trivially the condition in (6) is also fulfilled. The conditions **H1-H3** are the basic assumptions made for the stochastic volatility model (2)-(3) and they will be assumed to hold throughout. In addition for some of the moments of the return process we will need the following two conditions

H4. $\int_{\mathbb{R}_0^n} g^3(\mathbf{x})G(d\mathbf{x}) < \infty$,

H5. $\int_{\mathbb{R}_0^n} g^4(\mathbf{x})G(d\mathbf{x}) < \infty$.

I continue with fixing some notation associated with the jump-diffusion JDSV model, which will be used throughout. I denote the return over the period $(t, t+a]$ with

$$r_a(t) = p(t+a) - p(t). \tag{7}$$

The quadratic variation of the price process over the period $(t, t+a]$ is given by

$$[p,p]_{(t,t+a]} = \int_t^{t+a} \sigma^2(s)ds + \int_t^{t+a}\int_{R_0^n} g^2(\mathbf{x})\mu(ds, d\mathbf{x}), \tag{8}$$

and its predicted version (see Jacod and Shiryaev (2003) for a definition) is

$$\langle p,p\rangle_{(t,t+a]} = \int_t^{t+a} \sigma^2(s)ds + a\int_{R_0^n} g^2(\mathbf{x})G(d\mathbf{x}). \tag{9}$$

Note that the second term in (9) is nonstochastic since the jump measure is a Poisson random measure.

Also I denote the integrated variance over the period $(t, t+a]$ with

$$IV_a(t) = \int_t^{t+a} \sigma^2(s)ds. \tag{10}$$

This is a bit loose notation since the expression in (10) is just the integrated variance of the diffusion component in the price, that is the first component in (8). The jumps in the price also contribute for the integrated variance of the return (the second component in (8)), but because of the assumption in the model, this variance is not time varying. The integrated variance in (10) can be also written as

$$IV_a(t) = \int_{-\infty}^{t+a} H^a(t,s)k(\mathbf{x})\mu(ds, d\mathbf{x}), \tag{11}$$

where

$$H^a(t,s) = \begin{cases} \int_t^{t+a} f(z-s)dz & \text{if } s < t \\ \int_s^{t+a} f(z-s)dz & \text{if } t \leq s < t+a. \end{cases} \tag{12}$$

The result follows from a straightforward application of the Fubini theorem. Note that the stochastic variance in (3) is defined as a pathwise integral, which is almost surely finite.

The model provides analytical tractability. In the next few theorems I summarize moments related with the model, which will be used in the empirical part of the paper.

I start with deriving the conditional expectation and the first two moments of the integrated variance. The results are stated in the next theorem.

Theorem 1 (Moments of the Integrated Variance)

In the stochastic volatility model (2)-(3) assume conditions **H2** *and* **H3** *hold. Then for any* $s \leq t$ *we have*

$$\mathbb{E}_s(IV_a(t)) = \int_{-\infty}^{s} \int_{\mathbb{R}_0^n} H^a(t,u)k(\mathbf{x})\tilde{\mu}(du, d\mathbf{x}) + a\int_0^{\infty} f(s)ds \int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x}), \tag{13}$$

$$\mathbb{E}(IV_a(t)) = a\int_0^{\infty} f(s)ds \int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x}), \tag{14}$$

$$Var(IV_a(t)) = \int_{-\infty}^{a} (H^a(0,u))^2 du \int_{\mathbb{R}_0^n} k^2(\mathbf{x})G(d\mathbf{x}). \tag{15}$$

From equation (13) one can see that for a general form of the memory function $f(\cdot)$ we need all the past information in order to make a prediction for the integrated variance. However this information is not available to the econometrician and therefore such optimal forecast is infeasible.

In the next theorem I derive the first four centered moments of the return process, which can be used in estimation.

Theorem 2 (Unconditional Moments of the Returns Process)

For the stochastic volatility model (2)-(3) assume that conditions **H1**-**H5** *are satisfied. Then we have*

$$\mathbb{E}(r^a(t)) = a\alpha, \tag{16}$$

$$Var(r^a(t)) = a\int_0^\infty f(s)ds \int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x}) + a\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}), \tag{17}$$

$$\mathbb{E}(r^a(t) - \mathbb{E}(r^a(t)))^3 = 3\int_0^a H^a(0,u)du \int_{\mathbb{R}_0^n} k(\mathbf{x})g(\mathbf{x})G(d\mathbf{x}) + a\int_{\mathbb{R}_0^n} g^3(\mathbf{x})G(d\mathbf{x}), \tag{18}$$

$$\begin{aligned}\mathbb{E}(r^a(t) - \mathbb{E}(r^a(t)))^4 =& 3\int_{-\infty}^a (H^a(0,u))^2 du \int_{\mathbb{R}_0^n} k^2(\mathbf{x})G(d\mathbf{x}) + 3\Big(a\int_0^\infty f(s)ds \int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x})\Big)^2 \\ &+ a\int_{\mathbb{R}_0^n} g^4(\mathbf{x})G(d\mathbf{x}) + 6a^2\int_0^\infty f(s)ds \int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x}) \int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}) \\ &+ 6\int_0^a H^a(0,u)du \int_{\mathbb{R}_0^n} g^2(\mathbf{x})k(\mathbf{x})G(d\mathbf{x}) + 3a^2\Big(\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x})\Big)^2.\end{aligned} \tag{19}$$

Equation (17) shows that the variance of the return process has two parts. The first is the one due to the diffusion component and the second one is due to the jump component.

Equation (18) reveals the source of the skewness in the returns in this model. The first term in (18) is from the presence of leverage effect, which in this model reduces to negative linear relationship between the jumps in the price and the jumps in the stochastic variance. In addition skewness in the returns could be generated through skewness in the jump component of the price, which is the second component of equation (18).

Next I look at the covariance between the squared demeaned returns. The results are summarized in the following theorem

Theorem 3 (Covariance of the squared demeaned returns)

For the stochastic volatility model (2)-(3) assume conditions **H1**-**H3** *and condition* **H5** *hold. Then for $h = a, 2a, 3a....$ we have*

$$\begin{aligned}&Cov((r_a(0) - \mathbb{E}(r_a(0)))^2, (r_a(h) - \mathbb{E}(r_a(h)))^2) = \\ &\int_0^a H^a(h,u)du \int_{\mathbb{R}_0^n} g^2(\mathbf{x})k(\mathbf{x})G(d\mathbf{x}) + \int_{-\infty}^a H^a(h,u)H^a(0,u)du \int_{\mathbb{R}_0^n} k^2(\mathbf{x})G(d\mathbf{x}).\end{aligned} \tag{20}$$

The first term in equation (20) is due to the link between jumps in the price and those in the variance. If the jumps in price and the variance are independent, then this term will disappear. The formula for the covariance of the demeaned squared returns captures the memory of the stochastic process, and since it involves observable variables it could be used for making inference about the memory of the stochastic variance.

The following theorem gives the covariance between the third powers of the demeaned returns.

Theorem 4 (Covariance of the third powers of the demeaned returns)

For the stochastic volatility model (2)-(3) assume conditions **H1**-**H3** *and condition* **H4** *hold. Then for* $h = a, 2a, 3a....$ *we have*

$$Cov((r_a(0) - \mathbb{E}(r_a(0)))^3, (r_a(h) - \mathbb{E}(r_a(h)))^3) = 0. \tag{21}$$

The theorem implies that the autocorrelation of the demeaned returns raised to third power will be zero. This follows from the fact that the stochastic variance in this model is driven by Poisson random measure and therefore is independent from the Brownian motion in the price process. This on its turn implies that the continuous component in the price is conditionally Gaussian. Therefore correlation between the returns raised to the power three in this model can be generated solely through the price jumps. However in this class of jump driven stochastic volatility models the price jumps are time homogenous and this gives the result in (21).

I conclude this Section with stating a result associated with the leverage effect in the model.

Theorem 5 (Moments capturing the leverage effect)

For the stochastic volatility model (2)-(3) under conditions **H1**-**H3** *and* $h = a, 2a, 3a....$ *we have*

$$\mathbb{E}[(r_a(0) - \mathbb{E}(r_a(0)))^2(r_a(h) - \mathbb{E}(r_a(h)))] = 0, \tag{22}$$

$$\mathbb{E}[(r_a(0) - \mathbb{E}(r_a(0)))(r_a(h) - \mathbb{E}(r_a(h)))^2] = \int_0^a H^a(h,s)ds \int_{\mathbb{R}_0^n} k(\mathbf{x})g(\mathbf{x})G(d\mathbf{x}). \tag{23}$$

Equation (22) shows that in the stochastic volatility model (2)-(3) past squared demeaned returns cannot predict future returns. At the same time from equation (23) we see that in this model past returns can be used to predict future variance of the return process. This is consistent with the presence of a dynamic leverage effect observed in stock market data (see Bollerslev, Litvinova, and Tauchen (2005) and Tauchen (2005)) and here in this model it will exist iff $\int_{\mathbb{R}_0^n} k(\mathbf{x})g(\mathbf{x})G(d\mathbf{x}) \neq 0$, i.e there is leverage effect. Therefore these moments could be used for identification of the parameters controlling the leverage effect.

3 Pure Jump JDSV Model

The stochastic volatility model (2)-(3) is within the class of the traditional stochastic volatility models where the diffusive component of the price captures much of the variation and jumps are used to generate rare events (although here jumps could exhibit even infinite variation, due to the way of modeling of the jumps in the price - as an integral with respect to a compensated integer valued random measure). An alternative approach is to model prices as being driven solely by jumps (see Carr, Geman, Madan, and Yor (2002, 2003) among others). The idea behind such modelling is that very active pure jump processes will make unnecessary a continuous component in the price. Very active jump processes could generate both large shocks as well as small movements. In this Section I introduce the pure jump JDSV model which, as its name says, models the price as pure jump process. I also derive moments associated with this model, which will be used in the empirical part for its estimation.

The model is given by the following two equations

$$p(t) = p(0) + \alpha t + \int_0^t \int_{\mathbb{R}_0^n} \sigma(s-) g(\mathbf{x}) \tilde{\mu}(ds, d\mathbf{x}), \tag{24}$$

$$\sigma^2(t) = \int_{-\infty}^t \int_{\mathbb{R}_0^n} f(t-s) k(\mathbf{x}) \mu(ds, d\mathbf{x}), \tag{25}$$

where as in the jump-diffusion JDSV model, μ is a homogenous Poisson random measure with compensator ν, such that $\nu(ds, d\mathbf{x}) = dsG(d\mathbf{x})$. The price here is assumed to be a pure jump square integrable martingale (the conditions for its existence specified below) plus a constant drift term. The jumps in the price can exhibit paths of infinite variation. The state variable $\sigma(t)$ plays the same role here as the stochastic variance in the jump-diffusion JDSV model. Therefore the arguments in favor of modeling the stochastic variance in the jump driven stochastic volatility model (2)-(3) as a moving average of positive jumps apply here as well.

Unlike the jump-diffusion JDSV model, the pure jump JDSV model has jumps in the price which exhibit time variation

$$\Delta p(t) = \sigma(t-) g(\mathbf{x}) \qquad \text{and} \qquad \Delta \sigma^2(t) = f(0) k(\mathbf{x}).$$

The time variation in the jumps in the price is captured by the state variable $\sigma(t)$ (recall that the measure μ has no time variation in it). An alternative way of introducing time variation in the jumps is to time change a pure jump Lévy process by a subordinator. This approach was recently advanced in financial economics by Carr, Geman, Madan, and Yor (2003) and econometrically analyzed in Barndorff-Nielsen and Shephard (2005) under the assumption of independence of the time change from the Lévy process that is time changed.

The COGARCH model of Klüppelberg, Lindner, and Maller (2004) without continuous component in the price and its higher order extension defined in Brockwell, Chadraa, and Lindner (2004) are very similar to the pure jump JDSV models in (24)-(25) (for particular choice of the functions $f(\cdot)$, $g(\cdot)$ and $k(\cdot)$). However the COGARCH model is not nested in the pure jump JDSV models analyzed in this Section. In the empirical part I further elaborate on the similarity of these two modeling approaches.

Conditions exactly the same as **H1-H3** are assumed to hold here as well

S1. $\int_{\mathbb{R}_0^n} g^2(\mathbf{x}) G(d\mathbf{x}) < \infty$,

S2. $\int_0^\infty f(s) ds < \infty$ and $\int_{\mathbb{R}_0^n} k(\mathbf{x}) G(d\mathbf{x}) < \infty$,

S3. $\int_0^\infty f^2(s) ds < \infty$ and $\int_{\mathbb{R}_0^n} k^2(\mathbf{x}) G(d\mathbf{x}) < \infty$.

As in the jump-diffusion JDSV model with time-homogenous, these conditions guarantee that the return process and $\sigma^2(t)$ are covariance stationary. In addition for deriving some of the moments of the return process we will need the following two conditions

S4. $\int_{\mathbb{R}_0^n} g^4(\mathbf{x})G(d\mathbf{x}) < \infty$,

S5. $\int_{\mathbb{R}_0^n} g(\mathbf{x})k(\mathbf{x})G(d\mathbf{x}) = 0$ and $\int_{\mathbb{R}_0^n} g^3(\mathbf{x})G(d\mathbf{x}) = 0$.

Condition **S4** is needed for deriving the forth moment of the returns as well as for the covariance of the squared returns and it guarantees the finiteness of these moments. Condition **S5** on the other hand rules out linear dependence between the jumps and the variance and it is used in deriving closed form analytical expression for the forth moment of the returns and covariance of the squared returns. Therefore condition **S5** is restrictive as it rules out leverage effect in the model. On the other hand the assumptions **S1-S4** are not restrictive in the sense that they are the minimal assumptions needed to estimate the model by a method of moments type estimator.

The quadratic variation of the price process over the period $(t, t+a]$ is given by

$$[p,p]_{(t,t+a]} = \int_t^{t+a} \int_{\mathbb{R}_0^n} \sigma^2(s-)g^2(\mathbf{x})\mu(ds, d\mathbf{x}), \tag{26}$$

and its predicted version is

$$\langle p,p \rangle_{(t,t+a]} = \int_t^{t+a} \sigma^2(s)ds \int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}). \tag{27}$$

Note that unlike the jump-diffusion JDSV model, here the quadratic variation of the price process is a pure jump process. Its predicted version is proportional to the integrated variance of the jump driven stochastic volatility model in (10). Therefore here for convenience I preserve also the notation $H^a(t,s)$ as in equation (12).

Next I summarize moments of the return process, which will be used later for the estimation of the pure stochastic jump model in (24)-(25).

Theorem 6 *For the pure jump JDSV model (24)-(25), assume conditions* **S1**-**S5** *hold. Then we have*

$$\mathbb{E}(r_a(t)) = a\alpha, \tag{28}$$

$$Var(r_a(t)) = a\int_0^\infty f(s)ds \int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}) \int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x}), \tag{29}$$

$$
\begin{aligned}
E(r^a(t)-\mathbb{E}(r_a(t)))^4 =& a\int_0^\infty f^2(s)ds\int_{\mathbb{R}_0^n}k^2(\mathbf{x})G(d\mathbf{x})\int_{\mathbb{R}_0^n}g^4(\mathbf{x})G(d\mathbf{x})\\
&+a\Big(\int_0^\infty f(s)ds\int_{\mathbb{R}_0^n}k(\mathbf{x})G(d\mathbf{x})\Big)^2\int_{\mathbb{R}_0^n}g^4(\mathbf{x})G(d\mathbf{x})\\
&+6\int_0^a\int_0^s f(s-u)duds\int_0^\infty f(s)ds\int_{\mathbb{R}_0^n}g^2(\mathbf{x})G(d\mathbf{x})\int_{\mathbb{R}_0^n}k(\mathbf{x})G(d\mathbf{x})\int_{\mathbb{R}_0^n}g^2(\mathbf{x})k(\mathbf{x})G(d\mathbf{x})\\
&+3a^2\Big(\int_0^\infty f(s)ds\int_{\mathbb{R}_0^n}g^2(\mathbf{x})G(d\mathbf{x})\int_{\mathbb{R}_0^n}k(\mathbf{x})G(d\mathbf{x})\Big)^2\\
&+6\int_0^a\int_{-\infty}^u H^u(0,s)f(u-s)dsdu\int_{\mathbb{R}_0^n}k^2(\mathbf{x})G(d\mathbf{x})\Big(\int_{\mathbb{R}_0^n}g^2(\mathbf{x})G(d\mathbf{x})\Big)^2,
\end{aligned}
\tag{30}
$$

and for $h=a,2a,...$

$$
\begin{aligned}
Cov(r_a^2(t),r_a^2(t+h)) =& \int_{-\infty}^a H^a(h,u)H^a(0,u)du\int_{\mathbb{R}_0^n}k^2(\mathbf{x})G(d\mathbf{x})\Big(\int_{\mathbb{R}_0^n}g^2(\mathbf{x})G(d\mathbf{x})\Big)^2\\
&+\int_0^\infty f(s)ds\int_0^a H^a(h,u)du\int_{\mathbb{R}_0^n}k(\mathbf{x})G(d\mathbf{x})\int_{\mathbb{R}_0^n}g^2(\mathbf{x})G(d\mathbf{x})\int_{\mathbb{R}_0^n}g^2(\mathbf{x})k(\mathbf{x})G(d\mathbf{x}).
\end{aligned}
\tag{31}
$$

Condition **S5** rules out leverage, in the sense of linear relationship between the innovations in the price and in the state variable $\sigma^2(t)$. Therefore here I do not derive moments, which capture leverage effect as it was done for the jump-diffusion JDSV model. However note that condition **S5** does not preclude dependence between the jumps in $p(t)$ and $\sigma^2(t)$, it only rules out linear dependence.

4 Empirical Application

4.1 Power Variation Statistics and Estimation

In this section I define the power variation statistics used in this paper, which could be potentially applied as a way to disentangle jumps from continuous processes as well as to enhance the efficiency of the parameter estimation. I start with recalling the definition of p-th variation. For a general real valued function $h(\cdot)$ on $[0,T]$ denote with

$$s_h(p,\Delta)=\sum_{i=1}^n |h(t_{i+1})-h(t_i)|^p,$$

where $\Delta=\{t_i, i=1,...n\}$ such that $0=t_1<t_2...<t_n=T$ is an arbitrary partition of the interval $[0,T]$. Then the p-th variation of the function $h(\cdot)$ on the interval $[0,T]$ is defined as

$$Var_p(h,[0,T])=\sup\{s_h(p,\Delta):\Delta \quad \text{is a partition of} \quad [0,T]\}. \tag{32}$$

Note that the p-th power variation defined here is a particular case of the generalized multipower variation statistics defined in Barndorff-Nielsen, Graversen, Jacod, Podolskij, and Shephard (2005).

It is well known that for nontrivial continuous martingales the quadratic variation is almost surely finite, while p-th variation for $p > 2$ is 0, and for $0 < p < 2$ is infinity.

On the other hand for general discontinuous semimartingales the finiteness of the p-th variation is determined by the so called generalized Blumenthal-Getoor index (see Blumenthal and Getoor (1961) for the original definition and Woerner (2002) for its generalization), given by

$$\beta = \inf\{\tau : \int_0^t \int_{\mathbb{R}_0} (1 \wedge |x|^\tau)\nu(ds, dx) \in \mathcal{A}_{loc}\},$$

where $\nu(dt, dx)$ is the compensator of the jump measure of the semimartingale and $\mathcal{A}_{loc}$ is the space of locally integrable processes [1]. Therefore for $p \geq \beta$ the p-th variation of a jump semimartingale will converge to a finite number while for $p > 2$ and continuous semimartingale the p-th variation will converge to zero. This suggests that p-th variation statistics for $p > 2$ will single out the jump component of the price when using high frequency returns.

Based on these theoretical results I proceed with introducing the power variation statistics used in this study. I start with realized daily variance. It has been extensively studied in the finance literature over the past decade (see Andersen, Bollerslev, and Diebold (2005a) for a survey). The realized variance over day t with M intraday returns of length δ is denoted as $RV_\delta^M(t)$. I work with the convention that the unit of measurement is in days and therefore $1 = M\delta$. The formula for the daily realized variance is

$$RV_\delta^M(t) = \sum_{k=0}^{M-1} r_\delta^2(t + k\delta). \tag{33}$$

Since the log price process $p(t)$ is a semimartingale, the realized variance is a consistent estimator of the quadratic variation. Unlike the integrated variance, the realized daily variance is observable and therefore we can make inference based on its moments.

Analogously to the realized daily variance I introduce here another statistic, which also aggregate the high frequency data into a daily statistic. This is the realized forth power variation (denoted hereafter as FV). It is defined over a day t as

$$FV_\delta^M(t) = \sum_{k=0}^{M-1} r_\delta^4(t + k\delta). \tag{34}$$

As argued above for $p > 2$ we will capture only the jumps, while for $p = 2$ we will capture both the continuous and the jump component. Therefore, based on results in Barndorff-Nielsen and Shephard (2004), Barndorff-Nielsen, Shephard, and Winkel (2005) and Woerner (2002) among others, we have for the jump-diffusion JDSV model in (2)-(3)

$$RV_\delta^M(t) \overset{a.s.}{\rightarrow} \int_t^{t+M\delta} \sigma^2(s)ds + \int_t^{t+M\delta} \int_{\mathbb{R}_0^n} g^2(\mathbf{x})\mu(ds, d\mathbf{x}),$$

[1] Recall that a locally integrable process is a process for which there exists a sequence of increasing stopping times going to infinity almost surely, such that the stopped process is integrable.

$$FV_\delta^M(t) \overset{a.s.}{\to} \int_t^{t+M\delta} \int_{\mathbb{R}_0^n} g^4(\mathbf{x})\mu(ds, d\mathbf{x}).$$

Consequently, provided there are no jumps in the price FV will go to zero. In case of jumps however the FV statistic will not converge to zero. This suggests that this statistic could be used for separating jumps from continuous martingale components in the price.

By analogy for the pure jump JDSV model in (24)-(25) we have

$$RV_\delta^M(t) \overset{a.s.}{\to} \int_t^{t+M\delta} \int_{\mathbb{R}_0^n} \sigma^2(s-)g^2(\mathbf{x})\mu(ds, d\mathbf{x}),$$

$$FV_\delta^M(t) \overset{a.s.}{\to} \int_t^{t+M\delta} \int_{\mathbb{R}_0^n} \sigma^4(s-)g^4(\mathbf{x})\mu(ds, d\mathbf{x}).$$

Since in this model the price is comprised of jumps only, all of the statistics converge to the sum of the jumps raised to the appropriate power. In particular we do not have a component, which appears only in the Realized Variance as it is the case if a continuous component is present in the price.

Based on the results of the previous sections it is easy to derive certain moment conditions corresponding for the power variation statistics. Because of the properties of the power variation statistics discussed above I will be working in the estimation of the models with moments computed for RV and FV. A simulation based estimation could also take into account moments of other power (possibly noninteger) variation statistics (and not just the second and forth as used here), for which closed form analytical expressions are not available. This could potentially provide gain in efficiency in the estimation at least asymptotically (see the discussion along these lines in Ait-Sahalia (2004) in a time homogenous setting).

In the estimation of the JDSV models I set α to zero and fit it to demeaned stock market return data. This is not a serious limitation since in the models the mean is constant and the estimation is applied to high frequency data.

For the estimation of the stochastic volatility models I use a minimum distance type estimator and match the following statistics

- Mean of RV
- Autocorrelation of RV
- Variance of RV
- Mean of FV
- Forth moment of the daily returns

These statistics provide good identifying conditions for the different parameters of the model. For the autocorrelation of the RV I use lags one, four, seven, ten, thirteen, sixteen and nineteen as well as the sum of the autocorrelations from lag twenty till lag forty. Thus altogether I end up with twelve conditions. Adding more conditions could increase the asymptotic efficiency of the estimator, but in a finite sample this is also associated with a lower precision in estimating the optimal weighting matrix (see the Monte Carlo evidence in Andersen and Sørensen (1996)).

For the optimal weighting matrix I use a HAC estimator of the covariance matrix with a Bartlett kernel and a lag length of eighty. The estimation is performed using the MCMC approach of Chernozhukov and Hong (2003) of treating the Laplace transform of the objective function as an unnormalized likelihood function and applying MCMC to the pseudo posterior. Then the point estimates are the resulting mode of the pseudo posterior. The standard errors of the parameter estimates are the standard errors of the posterior distribution of the parameters, since the estimator used here is efficient. Further details about the estimation technique are relegated to Appendix E.

4.2 Modelling the memory of the stochastic volatility

For the estimation of the JDSV models introduced in the previous sections, the memory function in (3) and (25) respectively needs to be specified. Here I work with CARMA(2,1) kernel with distinct real autoregressive roots. The reason for the choice of CARMA(2,1) kernel is the good fit which two factor stochastic volatility models provide (see Alizadeh, Brandt, and Diebold (2002) and Chernov, Gallant, Ghysels, and Tauchen (2003) among others). The analogue of a two factor model in the setting here is the CARMA(2,1) kernel. It has the advantage over the multi factor modelling of being able to succinctly model the memory of the stochastic variance and at the same time provide even richer autocorrelation structures. Based on the studies mentioned above we expect that one of the autoregressive roots will be slowly mean reverting, corresponding to a persistent factor in the variance. The second autoregressive root is expected to be fast mean revering, which will correspond to the less persistent factor in the variance.

The CARMA(2,1) kernel with distinct real autoregressive roots is given by (see Brockwell (2001b) for details)

$$f(u) = \frac{b_0 + b_1\rho_1}{\rho_1 - \rho_2} e^{\rho_1 u} + \frac{b_0 + b_1\rho_2}{\rho_2 - \rho_1} e^{\rho_2 u}, \qquad u \geq 0. \tag{35}$$

For identification purposes I make the normalization $b_1 = 1$, which is convenient since it yields $f(0) = 1$, that is the weight of a shock in the variance occurring at the current moment is 1. Thus finally the kernel I am working with is

$$f(u) = \frac{b_0 + \rho_1}{\rho_1 - \rho_2} e^{\rho_1 u} + \frac{b_0 + \rho_2}{\rho_2 - \rho_1} e^{\rho_2 u}, \qquad u \geq 0. \tag{36}$$

For this choice of $f(\cdot)$ the kernel of the integrated variance $H^a(t,s)$ in (12) becomes

$$H^a(t,s) = \begin{cases} \frac{b_0+\rho_1}{\rho_1-\rho_2}\frac{e^{\rho_1 a}-1}{\rho_1}e^{\rho_1(t-s)} + \frac{b_0+\rho_2}{\rho_2-\rho_1}\frac{e^{\rho_2 a}-1}{\rho_2}e^{\rho_2(t-s)} & \text{if } s < t \\ \\ \frac{b_0+\rho_1}{\rho_1-\rho_2}\frac{e^{\rho_1(t-s+a)}-1}{\rho_1} + \frac{b_0+\rho_2}{\rho_2-\rho_1}\frac{e^{\rho_2(t-s+a)}-1}{\rho_2} & \text{if } t \leq s < t+a. \end{cases} \tag{37}$$

The necessary and sufficient condition for the CARMA(2,1) kernel to be nonnegative is $b_0 \geq -\max\{\rho_1, \rho_2\}$ (see Todorov and Tauchen (2005)). The CARMA(2,1) kernel in equation (36) reduces to a CARMA(1,0) kernel when $b_0 = -\min\{\rho_1, \rho_2\}$. Therefore the results for the CARMA(2,1) kernel could be specialized for the CARMA(1,0) case. In the empirical part I will estimate the JDSV models with CARMA(2,1) and CARMA(1,0) kernel. Note that the CARMA(1,0) choice for the kernel $f(\cdot)$ in the stochastic volatility model (2)-(3) corresponds to the case where the stochastic variance follows a Lévy-driven OU process as in Barndorff-Nielsen and Shephard (2001, 2002).

In Theorems 7 and 8 in Appendix C I derive the moments of RV and FV, which are used in the estimation of the two jump-driven stochastic volatility models (listed in Subsection 4.1) for the CARMA(2,1) specification of $f(\cdot)$.

4.3 Data Description

For the empirical application I use continuously compounded five-minute returns on the German Deutsch Mark/US Dollar (DM/$) spot exchange rate series. The data was kindly provided to the author by Tim Bollerslev. It spans the period from December 1, 1986 till June 30, 1999. From the data set were removed missing data, weekends, fixed holidays and similar calendar effects, with details explained in Andersen, Bollerslev, Diebold, and Labys (2001). The total number of days left in the data set is 3045, each of which conisists of 288 five minute continously componded returns.

In Table 1 I provide summary statistics for the daily returns as well as for the realized variance and realized forth power variation. Provided the exchange rate is a semimartingale, the mean of the squared daily returns and the mean of the realized variance should be approximately the same when the number of observations is large. However due to market microstructure noise the exchange rate might not be a semimartingale in which case such an equality does not need to hold. For the DM/$ exchange rate used here, these two statistics are approximately the same. This indirectly suggests that the market microstructure noise for the five-minute frequency does not have a big impact. This observation is further reinforced by the fact that the serial correlation in the high frequency returns is statistically insignificant.

Columns two and three in Table 1 show that the realized variance and realized forth power variation have fat tails. This particularly applies to the forth power variation. Being a proxy for the forth power of the jumps of the exchange rate over a trading day, the forth power variation will be close to zero in the days when there are no jumps and it will be relatively large in value in the days with jumps. That on its turn implies that the realized forth power variation will have significant mass in the tails. Figure 1 confirms this conjecture.

The bottom panel of Figure 1 shows the autocorrelation in the realized variance. Visual inspection of the autocorrelogram indicates that OU-type models will be unable to replicate it, an observation which is confirmed in the estimation results reported in the next Subsection.

4.4 Estimation Results

4.4.1 The Jump-Diffusion JDSV Model

In order to estimate the jump-diffusion JDSV model given in (2)-(3) with CARMA(2,1) choice for $f(\cdot)$, the functions $g(\mathbf{x})$ and $k(\mathbf{x})$, determining the jumps in the price and the variance, as well as the Poisson random measure μ need to be specified. Here I set $\mathbf{x}$ to be a scalar and make the following assumption

H6. $\quad g(\mathbf{x}) = \text{const1} \times x \qquad \text{and} \qquad k(\mathbf{x}) = \text{const2} \times x^2.$

This assumption means that the jumps in the variance are proportional to the squared price jumps. This jump specification bears analogy with the GARCH modeling in discrete time where the conditional variance is determined by the past squared returns. It is important also to note that this specification of the jumps does not rule out jumps of infinite variation in the price (which will be the case for example if the jumps in the price are proportional to the jumps in the variance).

Instead of specifying directly the Poisson random measure μ, given the assumption **H6**, I specify the following moments associated with the Poisson random measure

$$m_c = \int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x})\frac{b_0}{\rho_1\rho_2}, \tag{38}$$

$$m_d = \int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}), \tag{39}$$

$$v = \int_{\mathbb{R}_0^n} k^2(\mathbf{x})G(d\mathbf{x}). \tag{40}$$

The factor $\frac{b_0}{\rho_1\rho_2}$ in the expression for m_c is associated with the memory function, but makes m_c equal the variance of the continuous component of the price and thus easier to interpret. The expression in (39) is the variance of the discontinuous component. The expressions in (38)-(40) are associated with the cumulants [2] of the corresponding Lévy processes in the price and the

[2] Recall that the j-th cumulant of a random variable X (when it exists) is defined as

$$\kappa_j = \frac{\partial^j \log \mathbb{E}(e^{itX})}{\partial t^j}\bigg|_{t=0}, \qquad j = 1, 2...$$

variance. These cumulants of the Lévy processes are all we need to know for the Lévy measure μ in order to compute the moments used in the estimation of the jump-diffusion JDSV model via the minimum distance estimator specified in Subsection 4.1 (under the the jump specification **H6**).

The estimation results for the jump-diffusion JDSV model with CARMA(2,1) kernel and jump specification **H6** are reported in Table 2.

4.4.2 The Pure Jump JDSV Model

As for the jump-diffusion JDSV model I set $\mathbf{x}$ to be a scalar. I make the following assumptions for the functions $g(\mathbf{x})$ and $k(\mathbf{x})$ determining the jumps in the price and the variance and the measure μ in the pure jump JDSV model given in (24)-(25)

S6. $g(\mathbf{x}) = \text{const1} \times x$ and $k(\mathbf{x}) = \text{const2} \times x^2$.

S7. $\int_{\mathbb{R}_0} x^3 G(dx) = 0$.

As it was already discussed in Section 3 under such specification of the jumps in this model the (dynamic) leverage effect is ruled out. The assumption of no (dynamic) leverage effect is supported by the exchange rate data used here and therefore is not restrictive for this particular empirical application. Here the assumption **S6** allows for jumps in the price of infinite variation, which is particularly important in this context since the price is solely driven by jumps.

Under assumption **S6** we can see that the jumps in $\sigma^2(t)$ are proportional to the quadratic variation of the Lévy process driving the price. This is very similar to the COGARCH modeling (Klüppelberg, Lindner, and Maller (2004), Brockwell, Chadraa, and Lindner (2004)) where the jumps in $\sigma^2(t)$ are proportional to the quadratic variation of the discontinuous component of the price. What makes the pure jump JDSV model analyzed here different from the COGARCH intuitively is the fact that $\sigma^2(t)$ is an infinite moving average of the past squared Lévy jumps driving the price, while in the COGARCH it is a moving average of the past squared price jumps.

For the estimation of the pure jump JDSV model under the assumptions **S6** and **S7**, I parameterize the following expressions related with the driving Poisson random measure μ

$$m = \int_{\mathbb{R}_0^n} g^2(\mathbf{x}) G(d\mathbf{x}) \int_{\mathbb{R}_0^n} k(\mathbf{x}) G(d\mathbf{x}) \frac{b_0}{\rho_1 \rho_2}, \tag{41}$$

$$v = \sqrt{\int_{\mathbb{R}_0^n} g^4(\mathbf{x}) G(d\mathbf{x}) \int_{\mathbb{R}_0^n} k^2(\mathbf{x}) G(d\mathbf{x})}. \tag{42}$$

Similar to the jump-diffusion JDSV model the factor $\frac{b_0}{\rho_1 \rho_2}$ in m makes it equal to the variance of the return over a unit interval and thus is much easier to interpret. Expressions (41) and (42) are associated with cumulants of the Lévy process in the price and the variance and give all the information for μ needed to compute the moments used in the estimation of the pure jump JDSV model (given in Theorem 8) under the jump specification **S6-S7**.

The estimation results for the pure jump JDSV model with CARMA(2,1) kernel and jump specification **S6-S7** are summarized in Table 3.

4.4.3 A Two Factor Affine Jump Diffusion Model

To compare the performance of the JDSV models estimated above in this Subsection I report estimation results for a standard one and two factor affine jump-diffusion models. The estimation method is the same as for the estimation of the JDSV models and was discussed in Subsection 4.1. The two factor affine jump diffusion model nests the one factor affine jump diffusion model. It is found in the empirical finance literature (see for example Alizadeh, Brandt, and Diebold (2002), Andersen, Benzoni, and Lund (2002), Chernov, Gallant, Ghysels, and Tauchen (2003), Bollerslev and Zhou (2002), Jiang and Oomen (2004)) that two factor stochastic volatility models significantly outperform the one factor ones. As shown in Appendix D the realized variance in the two factor case has the same autocorrelation structure as the JDSV models analyzed here for the choice of the CARMA(2,1) kernel in (36). In the two factor affine jump diffusion stochastic volatility model the log price of the financial asset is given by

$$dp(t) = \alpha dt + \sqrt{V(t)}dW(t) + \int_{\mathbb{R}_0^n} g(\mathbf{x})\tilde{\mu}(ds, d\mathbf{x}). \tag{43}$$

In the estimation of the affine jump-diffusion models I set α to zero and fit the models to demeaned returns data like for the JDSV models. The function $g(\cdot)$ and the compensated random measure $\tilde{\mu}$ are as introduced in the JDSV models. $W(t)$ stands for a standard Brownian motion. The stochastic volatility $V(t)$ is a sum of two factors

$$V(t) = V_1(t) + V_2(t), \tag{44}$$

where

$$dV_1(t) = \kappa_1(\theta_1 - V_1(t))dt + \sigma_1\sqrt{V_1(t)}dB_1(t), \tag{45}$$

and

$$dV_2(t) = \kappa_2(\theta_2 - V_2(t))dt + \sigma_2\sqrt{V_2(t)}dB_2(t), \tag{46}$$

where $B_1(t)$ and $B_2(t)$ are two independent standard Brownian motions.

Like the jump-diffusion JDSV model, the jumps in the affine jump diffusion model (43)-(46) are time homogenous (of course in the general affine-jump diffusion models, the jumps could have time varying intensity (affine in the factors), see Duffie, Pan, and Singleton (2000), Filipović (2001) and Duffie, Filipović, and Schachermayer (2003) for details)

$$\Delta p(t) = g(\mathbf{x}) \qquad \text{and} \qquad \Delta V(t) = 0.$$

The two factors follow square root diffusion processes and take nonnegative values (provided certain restrictions on the parameters are imposed). In Theorem 9 in Appendix D I derive the moments used in the estimation (see also Bollerslev and Zhou (2002), Meddahi (2002) and Andersen, Bollerslev, and Meddahi (2004)). As in the pure jump JDSV model, the moments are derived under the assumptions of no leverage. In the affine jump diffusion analyzed here this reduces to $W(t)$, $B_1(t)$ and $B_2(t)$ being orthogonal to each other. This implies that the innovations in the price and in the volatility are independent, which is not in general true for the pure jump JDSV model under condition **S4**.

For the estimation of the affine jump diffusion model I parametrize the second and the forth cumulant of the jumps in the price that is I set

$$m = \int_{\mathbb{R}_0^n} g^2(\mathbf{x}) G(d\mathbf{x}), \tag{47}$$

$$v = \int_{\mathbb{R}_0^n} g^4(\mathbf{x}) G(d\mathbf{x}). \tag{48}$$

The results from the estimation of the affine jump diffusion model are reported in Table 4.

4.4.4 Discussion of the Results

The results in Tables 2-4 show that in almost all specifications the mean of the realized variance is pretty accurately matched. This fact is not surprising since the realized variance has little variation and as a result receive a significant weight in the estimator.

The first important question, which we try to answer using the estimation results, is whether price jumps matter for the ability of the models to fit the data. Looking at the results for the jump-diffusion JDSV model in Table 2 and the affine jump-diffusion model in Table 4 we can see that inclusion of jumps in the price significantly improves the fit of the model. This holds true regardless of the choice of the memory function for the jump-diffusion JDSV model and the number of factors in the affine jump-diffusion model. Also the big difference in the performance of the models with or without price jumps indicates that the moments used in the estimation significantly penalize for their omission. On the other hand we can see that overall the pure jump JDSV model under the jump specification given in **S6** and **S7** perform relatively bad. Therefore pure jump model in which the jumps in the variance are equal to the squares of the driving Lévy process in the price does not seem to provide good description of the data. At the same time the results in Table 2 indicate that such jump specification (given in **H6**) in which the jumps in the variance are squares of the Lévy jumps driving the price provides good fit for the jump-diffusion JDSV model, where a continuous component in the price is also present.

The second question of interest is how well is the persistence of the realized variance described by a CARMA(2,1) kernel. Comparing the performance of the JDSV models with CARMA(1,0) kernel with those with CARMA(2,1) kernel, we can see that the CARMA(2,1) kernel provides a much better fit for the autocorrelation structure of the Realized Variance. The same pattern emerges when comparing the one factor affine jump-diffusion model with a two factor one. This finding is in line with results in most of the empirical studies of multi-factor stochastic volatility models. What is important to note here is that the CARMA(2,1) kernel thus as good job as a two factor stochastic volatility model in capturing the persistence in the realized variance without the need of introducing multiple factors which are hard to interpret. On Figure 2 I compare the autocorrelation of the realized variance with that implied by the parameter estimates of the jump-diffusion JDSV model with CARMA(2,1) kernel, which contains price jumps. In the estimation, as already discussed in Subsection 4.1, I use only the first forty autocorrelations of the realized variance. Figure 2 shows that the CARMA(2,1) kernel provides pretty good fit and matches well the observed autocorrelations even beyond lag forty.

An implication of the CARMA(2,1) kernel for the JDSV models (jump-diffusion and pure jump) is that the realized variance has an ARMA(2,2) representation. The parameters of the ARMA(2,2) are functions of the structural parameters of the models and can be determined numerically. This is very convenient since it suggests that the realized variance could be put in a linear state space and thus the Kalman filter could be used to generate the conditional expectation of the realized variance, given the structural coefficients of the model. Similar result holds also in the superposition of Lévy-driven OU factors (and square root factors as well), which was used in Barndorff-Nielsen and Shephard (2002, 2005) to estimate parameters of Lévy-driven models via quasi maximum likelihood.

Finally comparing the results for the jump-diffusion JDSV models and the affine jump-diffusion models we see that the jump-diffusion JDSV models perform much better. In fact the only model, which successfully fits the statistics from the high frequency data used here (that is produces levels of the overidentification test well below commonly accepted critical values), is the jump-diffusion JDSV model with CARMA(2,1) memory function which contains jumps in the price specified in assumption **H6**. The parameter estimates of this model are reported in the last column of Table 2. The estimation results confirm our expectation regrading the autoregressive coefficients of the CARMA(2,1) kernel. The first autoregressive root has a half life of approximately thirty days and is therefore relatively slow mean reverting. On the other hand the second autoregressive root has a half life of an approximately half a day and thus has much faster mean reversion. Figure 2 illustrates that this kernel does provide a good fit of the empirical autocorrelation function. Turning to the parameters controlling the jumps we can see that the jump component in the price has around nine percent share in the total variance of the return process. This level is similar to the proportion of jumps found in financial asset prices using high frequency data in the studies of Andersen, Bollerslev, and Diebold (2005b) and Huang and Tauchen (2006) (these studies use nonparametric jump detection tests developed in Barndorff-Nielsen and Shephard (2004) to disentangle the jumps from the continuous component). Thus overall I find that the jump-driven JDSV model with CARMA(2,1) kernel, containing jumps in the price with specification given in **H6** is able to fit well the moments used in the estimation and captures the main empirical features observed in the data.

5 Conclusion

In this paper a general semiparametric jump-driven stochastic volatility model is introduced. This model has the distinctive feature that the state variables determining the time variation of the continuous and discontinuous component of the price, when time varying, are representable as moving averages of positive jumps. This allows for writing the integrated variance itself as another moving average of the same positive jumps. Using that I derive moments of the return process as well as power variation statistics, which aggregate on a daily level the data from the high-frequency returns. In the empirical part of the paper I model the memory of the realized variance using CARMA(2,1) kernel and specialize the jumps in the variance to be proportional to the square of the jumps driving the price. Under such specification I estimate a jump-diffusion and a pure jump JSDV model using high-frequency FX data and compare their performance with the traditional affine-jump diffusion models. Overall I find that the jump-diffusion model with CARMA(2,1) kernel performs best - better than the two factor affine jump-diffusion model. On the other hand the pure jump JDSV model with the particular jump specification used here is unable to fit the data. The empirical study confirms also the ability of the CARMA modeling to reproduce autocorrelation patterns of the traditionally used multi-factor stochastic volatility models.

Appendices

A Proof of Theorems 1-5

In the proofs of all theorems about the jump-diffusion JDSV model I use the following notation

$$X(t) = \int_0^t \sigma(s-)dW(s), \tag{A.1}$$

and

$$Y(t) = \int_0^t \int_{\mathbb{R}_0^n} g(\mathbf{x})\tilde{\mu}(ds, d\mathbf{x}). \tag{A.2}$$

Therefore over the period $(t, t+a]$, the return process is split into continuous and discontinuous martingale parts, denoted respectively as

$$X^a(t) = X(t+a) - X(t), \tag{A.3}$$

$$Y^a(t) = Y(t+a) - Y(t). \tag{A.4}$$

Therefore

$$r_a(t) = a\alpha + X^a(t) + Y^a(t). \tag{A.5}$$

Given the integrability conditions specified in equations **H1-H3** it is not hard to derive the joint characteristic function of arbitrary number of returns. This characteristic function could be used to prove all the theorems about moments of the return process. Computations are somewhat involved and therefore in the proofs I decided not to use the characteristic function.

A.1 Proof of Theorem 1

I start with deriving the first two moments of the spot variance $\sigma^2(t)$. Under condition **H2** we have

$$\mathbb{E}(\sigma^2(t)) = \mathbb{E}\bigg(\int_{-\infty}^t \int_{R_0^n} f(t-s)k(\mathbf{x})\mu(ds, d\mathbf{x})\bigg) = \int_0^\infty f(s)ds \int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x}). \tag{A.6}$$

For the mean of the squared spot variance under conditions **H2** and **H3** we have

$$\begin{aligned}\mathbb{E}(\sigma^4(t)) &= \mathbb{E}\bigg(\int_{-\infty}^t \int_{R_0^n} f(t-s)k(\mathbf{x})\tilde{\mu}(ds, d\mathbf{x}) + \int_0^\infty f(s)ds \int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x})\bigg)^2 \\ &= \int_0^\infty f^2(s)ds \int_{\mathbb{R}_0^n} k^2(\mathbf{x})G(d\mathbf{x}) + \bigg(\int_0^\infty f(s)ds \int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x})\bigg)^2.\end{aligned} \tag{A.7}$$

Then for the mean of the integrated variance we have

$$\mathbb{E}(IV_a(t)) = \mathbb{E}\bigg(\int_t^{t+a} \sigma^2(s)ds,\bigg) = a\mathbb{E}(\sigma^2(s)). \tag{A.8}$$

and (14) follows from (A.6).

Next I derive the result in (15) for the variance of the integrated variance. First note that for $s \leq t$

$$\mathbb{E}\bigg(\int_s^t \sigma^2(u)du\bigg)^2 \leq (t-s)\mathbb{E}\bigg(\int_s^t \sigma^4(u)du\bigg),$$

and therefore the finiteness of the second moment of the integrated variance follows from the conditions **H2** and **H3**. Then using the expression for the integrated variance in (11) we have

$$\begin{aligned}
\mathbb{E}(IV_a^2(t)) &= \mathbb{E}\bigg(\int_t^{t+a} \sigma^2(s)ds\bigg)^2 \\
&= \mathbb{E}\bigg(\int_{-\infty}^{a}\int_{\mathbb{R}_0^n} H^a(0,u)k(\mathbf{x})\tilde{\mu}(du,d\mathbf{x}) + \int_{-\infty}^{a} H^a(0,u)du\int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x})\bigg)^2 \\
&= \int_{-\infty}^{a}(H^a(0,u))^2du\int_{\mathbb{R}_0^n} k^2(\mathbf{x})G(d\mathbf{x}) + \bigg(\int_{-\infty}^{a} H^a(0,u)du\int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x})\bigg)^2 \\
&= \int_{-\infty}^{a}(H^a(0,u))^2du\int_{\mathbb{R}_0^n} k^2(\mathbf{x})G(d\mathbf{x}) + \bigg(a\int_0^\infty f(s)ds\int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x})\bigg)^2,
\end{aligned} \tag{A.9}$$

and from here using (14), the result for the variance of the integrated variance in (15) follows.

Finally I prove the claim in (13) for the conditional expectation of the integrated variance.

$$\begin{aligned}
\mathbb{E}_s(IV_a(t)) &= \mathbb{E}_s\bigg(\int_t^{t+a} \sigma^2(s)ds\bigg) \\
&= \mathbb{E}_s\bigg(\int_{-\infty}^{t+a}\int_{\mathbb{R}_0^n} H^a(t,u)k(\mathbf{x})\tilde{\mu}(du,d\mathbf{x}) + \int_{-\infty}^{t+a} H^a(t,u)du\int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x})\bigg) \\
&= \int_{-\infty}^{s}\int_{\mathbb{R}_0^n} H^a(t,u)k(\mathbf{x})\tilde{\mu}(du,d\mathbf{x}) + a\int_0^\infty f(s)ds\int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x}).
\end{aligned} \tag{A.10}$$

□

A.2 Proof of Theorem 2

Since the integrals $X^a(t)$ and $Y^a(t)$ are with respect to martingale measures we have

$$\mathbb{E}(X^a(0)) = 0, \tag{A.11}$$

and

$$\mathbb{E}(Y^a(0)) = \int_0^a\int_{\mathbb{R}_0^n} g(\mathbf{x})\tilde{\mu}(ds,d\mathbf{x}) = 0, \tag{A.12}$$

and from here the result for the mean in (16) follows.

To derive the variance of the returns, I derive the variance of its continuous and discontinuous parts. For the continuous part, using the integrability of the integrated variance established in Theorem 1 and Ito isometry we have

$$\mathrm{Var}(X^a(0)) = \mathbb{E}(X^a(0))^2 = \mathbb{E}\bigg(\int_0^a \sigma^2(s-)ds\bigg) = a\mathbb{E}(\sigma^2(s)), \tag{A.13}$$

For the variance of the jump component of the returns, I use condition **H1** and Ito isometry to get

$$\mathrm{Var}(Y^a(0)) = \mathbb{E}(Y^a(0))^2 = a\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}). \tag{A.14}$$

Using

$$\mathrm{Var}(r^a(0)) = \mathrm{Var}(X^a(0)) + \mathrm{Var}(Y^a(0)), \tag{A.15}$$

the result in (17) follows.

Now I derive the third central moment in (18). I make use of the fact that the filtration generated by the random measure μ is orthogonal to $W(t)$ and therefore conditional on it $X^a(t)$ is normal. This implies

$$\begin{aligned}\mathbb{E}(r^a(0) - \mathbb{E}(r^a(0)))^3 &= \mathbb{E}(X^a(0))^3 + 3\mathbb{E}(X^a(0)^2Y^a(0)) + 3\mathbb{E}(X^a(0)Y^a(0)^2) + \mathbb{E}(Y^a(0))^3\\ &= 3\mathbb{E}\Big(\int_0^a \sigma^2(s)ds\int_0^a\int_{\mathbb{R}_0^n} g(\mathbf{x})\tilde{\mu}(ds,d\mathbf{x})\Big) + a\int_{\mathbb{R}_0^n} g^3(\mathbf{x})G(d\mathbf{x}),\end{aligned} \tag{A.16}$$

where I made use of

$$\mathbb{E}(X^a(0))^3 = 0 \qquad \text{and} \qquad \mathbb{E}X^a(0)(Y^a(0)^2) = 0, \tag{A.17}$$

and provided condition **H4** is satisfied

$$\mathbb{E}(Y^a(0))^3 = a\int_{\mathbb{R}_0^n} g^3(\mathbf{x})G(d\mathbf{x}). \tag{A.18}$$

Further I simplify the first term in equation (A.16) by using the expression for the integrated variance in (11)

$$\begin{aligned}&\mathbb{E}\Big(\int_0^a \sigma^2(s)ds\int_0^a\int_{\mathbb{R}_0^n} g(\mathbf{x})\tilde{\mu}(ds,d\mathbf{x})\Big)\\ &= \mathbb{E}\Big[\Big(\int_{-\infty}^a\int_{\mathbb{R}_0^n} H^a(0,u)k(\mathbf{x})\tilde{\mu}(du,d\mathbf{x}) + \int_{-\infty}^a H^a(0,u)du\int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x})\Big)\int_0^a\int_{\mathbb{R}_0^n} g(\mathbf{x})\tilde{\mu}(ds,d\mathbf{x})\Big]\\ &= \int_0^a H^a(0,u)du\int_{\mathbb{R}_0^n} k(\mathbf{x})g(\mathbf{x})G(d\mathbf{x}),\end{aligned} \tag{A.19}$$

and the result in (18) follows.

Finally I derive the expression for the forth central moment in (19). Similar argument as the one used for the second and third central moments leads to

$$\begin{aligned}\mathbb{E}(r^a(0) - \mathbb{E}(r^a(0)))^4 &= \mathbb{E}(X^a(0))^4 + \mathbb{E}(Y^a(0))^4 + 6\mathbb{E}(X^a(0))^2(Y^a(0))^2\\ &= 3\mathbb{E}\Big(\int_0^a \sigma^2(s-)ds\Big)^2 + a^2\Big(\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x})\Big)^2\\ &\quad + a\int_{\mathbb{R}_0^n} g^4(\mathbf{x})G(d\mathbf{x})\\ &\quad + 6\mathbb{E}\Big[\Big(\int_0^a \sigma^2(s-)ds\Big)\Big(\int_0^a\int_{\mathbb{R}_0^n} g(\mathbf{x})\tilde{\mu}(ds,d\mathbf{x})\Big)^2\Big],\end{aligned} \tag{A.20}$$

where I made use of

$$\mathbb{E}(X^a(0))^4 = 3\mathbb{E}\bigg(\int_0^a \sigma^2(s-)ds\bigg)^2, \tag{A.21}$$

and (provided condition (19) is satisfied)

$$\mathbb{E}(Y^a(0))^4 = a\int_{\mathbb{R}_0^n} g^4(\mathbf{x})G(d\mathbf{x}) + a^2\bigg(\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x})\bigg)^2. \tag{A.22}$$

The first term in equation (A.20) is just the expected value of the squared integrated variance and consequently the expression in (15) could be used to simplify. Now I simplify the last term in (A.20).

By Ito's lemma we have

$$\begin{aligned} Y^2(t) &= \int_s^t 2Y(u-)dY(u) + \sum_{s<u\leq t}(Y^2(u) - Y^2(u-) - 2Y(u-)\Delta Y(u)) \\ &= \int_s^t 2Y(u-)dY(u) + \sum_{s<u\leq t}(\Delta Y(u))^2, \end{aligned} \tag{A.23}$$

from where using the integrability condition **H1** we have

$$Y^2(t) = \int_s^t \int_{\mathbb{R}_0^n}(2Y(u-)g(\mathbf{x}) + g^2(\mathbf{x}))\tilde{\mu}(du, d\mathbf{x}) + (t-s)\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}), \tag{A.24}$$

therefore provided the conditions **H1-H3** and **H5** are satisfied, using the square integrability of the integrated variance established in Theorem 1 the last term in (A.20) could be further written as

$$\begin{aligned} &\mathbb{E}\quad\bigg[\bigg(\int_0^a \sigma^2(s-)ds\bigg)\bigg(\int_0^a\int_{\mathbb{R}_0^n} g(\mathbf{x})\tilde{\mu}(ds, d\mathbf{x})\bigg)^2\bigg] \\ &= \quad\mathbb{E}\bigg[\bigg(\int_{-\infty}^a\int_{\mathbb{R}_0^n} H^a(0,u)k(\mathbf{x})\tilde{\mu}(du, d\mathbf{x}) + \int_{-\infty}^a H^a(0,u)du\int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x})\bigg)\bigg(\int_0^a\int_{\mathbb{R}_0^n}(2Y(s-)g(\mathbf{x}) \\ &\quad +g^2(\mathbf{x}))\tilde{\mu}(ds, d\mathbf{x}) + a\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x})\bigg)\bigg] \\ &= \quad a\int_{-\infty}^a H^a(0,u)du\int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x})\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}) + \int_0^a H^a(0,u)du\int_{\mathbb{R}_0^n} k(\mathbf{x})g^2(\mathbf{x})G(d\mathbf{x}), \end{aligned} \tag{A.25}$$

plugging this expression back in (A.20), together with the expression for the expected value of the square of the integrated variance we get (19). □

A.3 Proof of Theorem 3

I begin with

$$\begin{aligned} \mathbb{E}[(X^a(0) + Y^a(0))^2(X^a(h) + Y^a(h))^2] &= \mathbb{E}[((X^a(0))^2 + (Y^a(0))^2)((X^a(h))^2 + (Y^a(h))^2)] \\ &= \mathbb{E}((X^a(0))^2(X^a(h))^2) + \mathbb{E}((X^a(0))^2(Y^a(h))^2) \\ &\quad +\mathbb{E}((Y^a(0))^2(X^a(h))^2) + \mathbb{E}((Y^a(0))^2(Y^a(h))^2), \end{aligned} \tag{A.26}$$

and treat each of the four terms separately.

1. Expression for $\mathbb{E}((Y^a(0))^2(Y^a(h))^2)$.

Using the time homogeneity property of the Poisson random measure

$$\mathbb{E}((Y^a(0))^2(Y^a(h))^2) = \mathbb{E}(Y^a(0))^2\mathbb{E}(Y^a(h))^2 = a^2\Big(\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x})\Big)^2. \tag{A.27}$$

2. Expression for $\mathbb{E}((X^a(0))^2(X^a(h))^2)$.

By conditioning on the filtration generated by the spot variance, making use of the fact that the Brownian motion $W(t)$ is independent from the spot variance process and using the fact that $\sigma^2(s)$ has no fixed time of discontinuity we have

$$\begin{aligned}\mathbb{E}[(X^a(0))^2(X^a(h))^2] &= \mathbb{E}\Big[\Big(\int_0^a \sigma(s-)dW(s)\Big)^2\Big(\int_h^{h+a} \sigma(s-)dW(s)\Big)^2\Big] \\ &= \mathbb{E}\Big[\int_0^a \sigma^2(s-)ds\int_h^{h+a} \sigma^2(s-)ds\Big] \\ &= \mathbb{E}(IV_a(0)IV_a(h)).\end{aligned} \tag{A.28}$$

Therefore using the result in (13) for the conditional expectation of the integrated variance

$$\begin{aligned}&\mathbb{E}[(X^a(0))^2(X^a(h))^2] = \mathbb{E}\Big[\int_0^a \sigma^2(s-)ds\mathbb{E}_a\Big(\int_h^{h+a} \sigma^2(s-)ds\Big)\Big] \\ &= \mathbb{E}\Big[\int_0^a \sigma^2(s-)ds\Big(\int_{-\infty}^a\int_{\mathbb{R}_0^n} H^a(h,u)k(\mathbf{x})\tilde{\mu}(du,d\mathbf{x}) + a\int_0^\infty f(s)ds\int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x})\Big)\Big] \\ &= \mathbb{E}\Big[\int_0^a \sigma^2(s-)ds\Big(\int_{-\infty}^a\int_{\mathbb{R}_0^n} H^a(h,u)k(\mathbf{x})\tilde{\mu}(du,d\mathbf{x})\Big)\Big] + a^2\Big(\int_0^\infty f(s)ds\Big)^2\Big(\int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x})\Big)^2,\end{aligned} \tag{A.29}$$

where $\mathbb{E}_a(\cdot)$ stands for $\mathbb{E}(\cdot|\mathcal{F}_a)$.

With the help of the representation of the integrated variance as an integral with respect to the random measure μ given in (11) we finally have

$$\begin{aligned}\mathbb{E}[(X^a(0))^2(X^a(h))^2] \quad &= \quad \int_{-\infty}^a H^a(h,u)H^a(0,u)du\int_{\mathbb{R}_0^n} k^2(\mathbf{x})G(d\mathbf{x}) \\ &\quad +a^2\Big(\int_0^\infty f(s)ds\Big)^2\Big(\int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x})\Big)^2.\end{aligned} \tag{A.30}$$

From here

$$\mathrm{Cov}((X^a(0))^2,(X^a(h))^2) = \int_{-\infty}^a H^a(h,u)H^a(0,u)du\int_{\mathbb{R}_0^n} k^2(\mathbf{x})G(d\mathbf{x}). \tag{A.31}$$

3. Expression for $\mathbb{E}((X^a(0))^2(Y^a(h))^2)$.

Using the fact that the Brownian motion $W(t)$ is independent from the filtration generated by the random measure μ and the properties of the Poisson random measure we have

$$\begin{aligned}\mathbb{E}[(X^a(0))^2(Y^a(h))^2] &= \mathbb{E}\Big[\int_0^a \sigma^2(s-)ds(Y^a(h))^2\Big] \\ &= \mathbb{E}\Big[\int_0^a \sigma^2(s-)ds\Big(\int_h^{h+a}\int_{\mathbb{R}_0^n} g(\mathbf{x})\tilde{\mu}(ds,d\mathbf{x})\Big)^2\Big] \\ &= a^2\mathbb{E}(\sigma^2(s))\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}) \\ &= a^2\int_0^\infty f(s)ds\int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x})\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}).\end{aligned} \tag{A.32}$$

4. Expression for $\mathbb{E}((Y^a(0))^2(X^a(h))^2)$.

Using the expression in (13) for the conditional expected value of the integrated variance, conditions **H1-H3** and **H5**, as well as the square integrability of $Y(s)$ we have

$$\begin{aligned}\mathbb{E}[(Y^a(0))^2(X^a(h))^2] &= \mathbb{E}\Big[(Y^a(0))^2\int_h^{h+a}\sigma^2(s-)ds\Big] \\ &= \mathbb{E}\Big[(Y^a(0))^2\mathbb{E}_a\int_h^{h+a}\sigma^2(s-)ds\Big] \\ &= \mathbb{E}\Big[\int_0^a\int_{\mathbb{R}_0^n}(2Y(s-)g(\mathbf{x})+g^2(\mathbf{x}))\tilde{\mu}(ds,d\mathbf{x})\int_{-\infty}^a\int_{\mathbb{R}_0^n}H^a(h,u)k(\mathbf{x})\tilde{\mu}(du,d\mathbf{x})\Big] \\ &\quad +a\int_0^\infty f(s)ds\int_{\mathbb{R}_0^n}k(\mathbf{x})G(d\mathbf{x})\mathbb{E}\Big[\int_0^a\int_{\mathbb{R}_0^n}(2Y(s-)g(\mathbf{x})+g^2(\mathbf{x}))\mu(ds,d\mathbf{x})\Big] \\ &= \int_0^a H^a(h,u)du\int_{\mathbb{R}_0^n}g^2(\mathbf{x})k(\mathbf{x})G(d\mathbf{x}) \\ &\quad +a^2\int_0^\infty f(s)ds\int_{\mathbb{R}_0^n}g^2(\mathbf{x})G(d\mathbf{x})\int_{\mathbb{R}_0^n}k(\mathbf{x})G(d\mathbf{x}).\end{aligned} \tag{A.33}$$

Combining the expressions for $\mathbb{E}((Y^a(0))^2(Y^a(h))^2)$, $\mathbb{E}((X^a(0))^2(X^a(h))^2)$, $\mathbb{E}((X^a(0))^2(Y^a(h))^2)$, $\mathbb{E}((Y^a(0))^2(X^a(h))^2)$ into equation (A.26) we have

$$\begin{aligned}\mathbb{E}[(X^a(0)+Y^a(0))^2(X^a(h)+Y^a(h))^2] &= \int_0^a H^a(h,u)du\int_{\mathbb{R}_0^n}g^2(\mathbf{x})k(\mathbf{x})G(d\mathbf{x}) \\ &\quad +\int_{-\infty}^a H^a(h,u)H^a(0,u)du\int_{\mathbb{R}_0^n}k^2(\mathbf{x})G(d\mathbf{x}) \\ &\quad +a^2\Big(\int_{\mathbb{R}_0^n}g^2(\mathbf{x})G(d\mathbf{x})+\int_0^\infty f(s)ds\int_{\mathbb{R}_0^n}k(\mathbf{x})G(d\mathbf{x})\Big)^2.\end{aligned} \tag{A.34}$$

Using the fact that

$$\mathbb{E}(r_a(0)-\mathbb{E}(r_a(0))^2 = \mathbb{E}[(X^a(0)+Y^a(0))^2] = a\int_0^\infty f(s)ds\int_{\mathbb{R}_0^n}k(\mathbf{x})G(d\mathbf{x})+a\int_{\mathbb{R}_0^n}g^2(\mathbf{x})G(d\mathbf{x}), \tag{A.35}$$

we finally have

$$\operatorname{Cov}((r_a(0)-\mathbb{E}(r_a(0)))^2,(r_a(h)-\mathbb{E}(r_a(h)))^2)=\operatorname{Cov}[(X^a(0)+Y^a(0))^2,(X^a(h)+Y^a(h))^2]$$
$$=\mathbb{E}[((X^a(0))^2+(Y^a(0))^2)(X^a(h))^2+(Y^a(h))^2)]-\Big(\mathbb{E}(r_a(0)-\mathbb{E}(r_a(0)))^2\Big)^2. \tag{A.36}$$

and hence the result in (20) follows. □

A.4 Proof of Theorem 4

Using the independence of the Brownian motion $W(t)$ from the filtration created by the random measure μ we have

$$\mathbb{E}\Big((r_a(0)-\mathbb{E}(r_a(0)))^3(r_a(h)-\mathbb{E}(r_a(h)))^3\Big)=\mathbb{E}\Big((X^a(0)+Y^a(0))^3(X^a(h)+Y^a(h))^3\Big)$$
$$=\mathbb{E}[(Y^a(0)Y^a(h))^3]+3\mathbb{E}[(Y^a(0))^3Y^a(h)(X(h))^2]+3\mathbb{E}[(Y^a(h))^3Y^a(0)(X(0))^2]+9\mathbb{E}[(X(0))^2Y(0)(X(h))^2Y(h)]. \tag{A.37}$$

Therefore we are left with simplifying each of the terms in the last expression.

From the properties of the Poisson random measure μ and provided condition **H4** is satisfied

$$\mathbb{E}\Big(Y_a^3(0)Y_a^3(h)\Big)=a^2\Big(\int_{\mathbb{R}_0^n}g^3(\mathbf{x})G(d\mathbf{x})\Big)^2.$$

In order to simplify the last two terms in equation (A.37) I use that provided conditions **H1-H3** are satisfied, the following is true

$$\begin{aligned}\mathbb{E}_a\Big(X_a^2(h)Y_a(h)\Big)&=\mathbb{E}_a\Big(\int_{-\infty}^{h+a}\int_{\mathbb{R}_0^n}H^a(h,s)k(\mathbf{x})\mu(ds,d\mathbf{x})\int_h^{h+a}\int_{\mathbb{R}_0^n}g(\mathbf{x})\tilde{\mu}(ds,d\mathbf{x})\Big)\\&=\mathbb{E}\Big(\int_h^{h+a}\int_{\mathbb{R}_0^n}H^a(h,s)k(\mathbf{x})\mu(ds,d\mathbf{x})\int_h^{h+a}\int_{\mathbb{R}_0^n}g(\mathbf{x})\tilde{\mu}(ds,d\mathbf{x})\Big)\\&=\int_h^{h+a}H^a(h,s)ds\int_{\mathbb{R}_0^n}k(\mathbf{x})g(\mathbf{x})G(d\mathbf{x}).\end{aligned}$$

Therefore

$$\mathbb{E}\Big(X_a^2(0)Y_a(0)X_a^2(h)Y_a(h)\Big)=\Big(\int_0^a H^a(0,s)ds\int_{\mathbb{R}_0^n}k(\mathbf{x})g(\mathbf{x})G(d\mathbf{x})\Big)^2,$$

$$\mathbb{E}\Big(Y_a^3(0)X_a^2(h)Y_a(h)\Big)=a\int_{\mathbb{R}_0^n}g^3(\mathbf{x})G(d\mathbf{x})\Big(\int_0^a H^a(0,s)ds\int_{\mathbb{R}_0^n}k(\mathbf{x})g(\mathbf{x})G(d\mathbf{x})\Big),$$

$$\mathbb{E}\Big(Y_a^3(h)X_a^2(0)Y_a(0)\Big)=a\int_{\mathbb{R}_0^n}g^3(\mathbf{x})G(d\mathbf{x})\Big(\int_0^a H^a(0,s)ds\int_{\mathbb{R}_0^n}k(\mathbf{x})g(\mathbf{x})G(d\mathbf{x})\Big).$$

Combining everything together

$$\mathbb{E}\Big((r_a(0)-\mathbb{E}(r_a(0)))^3(r_a(h)-\mathbb{E}(r_a(h)))^3\Big)=9\Big(\int_0^a H^a(0,s)ds\int_{\mathbb{R}_0^n}k(\mathbf{x})g(\mathbf{x})G(d\mathbf{x})\Big)^2$$
$$+a^2\Big(\int_{\mathbb{R}_0^n}g^3(\mathbf{x})G(d\mathbf{x})\Big)^2+6a\int_{\mathbb{R}_0^n}g^3(\mathbf{x})G(d\mathbf{x})\Big(\int_0^a H^a(0,s)ds\int_{\mathbb{R}_0^n}k(\mathbf{x})g(\mathbf{x})G(d\mathbf{x})\Big),$$

from where the result in (21) follows. □

A.5 Proof of Theorem 5

Using the properties of the Poisson random measure and the fact that the Brownian motion $W(t)$ is independent from the filtration generated by the random measure μ we have

$$\mathbb{E}[(r_a(0)-\mathbb{E}(r_a(0)))^2(r_a(h)-\mathbb{E}(r_a(h)))] = \mathbb{E}[(X^a(0)+Y^a(0))^2(X^a(h)+Y^a(h))] = 0. \quad (A.38)$$

Using the same reasoning as above

$$\mathbb{E}[(r_a(0)-\mathbb{E}(r_a(0)))(r_a(h)-\mathbb{E}(r_a(h)))^2] = \mathbb{E}[(X^a(0)+Y^a(0))(X^a(h)+Y^a(h))^2] = \mathbb{E}[Y^a(0)(X^a(h))^2]. \quad (A.39)$$

Using the expression in (15) for the integrated variance, the square integrability of the integrated variance and conditions **H1-H3**, we have

$$\begin{aligned}\mathbb{E}[Y^a(0)(X^a(h))^2] &= \mathbb{E}\left[\int_0^a \int_{\mathbb{R}_0^n} g(\mathbf{x})\tilde{\mu}(ds, d\mathbf{x}) \int_{-\infty}^{h+a} H^a(h,s)k(\mathbf{x})\mu(ds,d\mathbf{x})\right] \\ &= \int_0^a H^a(h,s)ds \int_{\mathbb{R}_0^n} k(\mathbf{x})g(\mathbf{x})G(d\mathbf{x}),\end{aligned} \quad (A.40)$$

and from here the second claim in Theorem 5 follows. □

B Proof of Theorem 6

I introduce similar notation to the one used in the proofs of the theorems for the jump-diffusion JDSV model.

$$Y(t) = \int_0^t \int_{\mathbb{R}_0^n} \sigma(s-)g(\mathbf{x})\tilde{\mu}(ds, d\mathbf{x}), \quad (B.1)$$

and for the period $(t, t+a]$

$$Y^a(t) = Y(t+a) - Y(t),$$

therefore the return over the period $(t, t+a]$ is

$$r_a(t) = a\alpha + Y^a(t).$$

Under assumptions **S1** and **S2**, $Y(t)$ in (B.1) is a square integrable martingale. Before proceeding with the derivations of the moments in the theorem I establish that under the conditions **S1-S4** $\mathbb{E}(\sup_{s\le t} Y^4(t)) < \infty$ for every t. For this I use the Burkholder-Davis-Gundy inequality (see Protter (1990)), from which it is sufficient to establish the finiteness of $\mathbb{E}([Y,Y]_{[0,t]})^2$. The quadratic variation of the jump martingale $Y(t)$ is given by

$$[Y,Y]_{[0,t]} = \int_0^t \int_{\mathbb{R}_0^n} \sigma^2(s-)g^2(\mathbf{x})\mu(ds, d\mathbf{x}),$$

therefore using conditions **S1-S3**, combined with the result for the finiteness of the second moment of $\int_0^t \sigma^2(s)ds$ shown in Theorem 1 we have

$$\begin{aligned}
\mathbb{E}([Y,Y]_{[0,t]})^2 &= \mathbb{E}\Big(\int_0^t \int_{\mathbb{R}_0^n} \sigma^2(s-)g^2(\mathbf{x})\mu(ds,d\mathbf{x})\Big)^2 \\
&= \mathbb{E}\Big(\int_0^t \int_{\mathbb{R}_0^n} \sigma^2(s-)g^2(\mathbf{x})\tilde{\mu}(ds,d\mathbf{x}) + \int_0^t \sigma^2(s)ds \int_{\mathbf{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x})\Big)^2 \\
&= \mathbb{E}\Big(\int_0^t \sigma^4(s-)ds\Big)\int_{\mathbb{R}_0^n} g^4(\mathbf{x})G(d\mathbf{x}) + \Big(\mathbb{E}\Big(\int_0^t \sigma^2(s)ds\Big)\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x})\Big)^2.
\end{aligned}$$

This establishes the claim.

I proceed with deriving the moments of the return process.

$$\mathbb{E}(r_a(0)) = a\alpha + \mathbb{E}(Y(a)) = a\alpha.$$

For the variance of the returns I make use of the fact that $Y(t)$ is a square integrable martingale.

$$\begin{aligned}
\mathrm{Var}(r_a(0)) &= E\Big(\int_0^a \int_{\mathbb{R}_0^n} \sigma(s-)g(\mathbf{x})\tilde{\mu}(ds,d\mathbf{x})\Big)^2 \\
&= E\Big(\int_0^a \sigma^2(s)ds\Big)\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}) \\
&= aE(\sigma^2(s))\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}),
\end{aligned}$$

and the result in (29) for the variance of the returns follows from the result in (A.6).

I prove now the result for the forth central moment of the returns in equation (30).

$$E(r_a(0) - \mathbb{E}(r_a(0)))^4 = E(Y(a))^4.$$

By Ito's lemma

$$Y^4(t) = Y^4(s) + \int_s^t 4Y^3(u-)dY(u) + \sum_{s<u\le t}\Big((\Delta Y(u))^4 + 4Y(u-)(\Delta Y(u))^3 + 6Y^2(u-)(\Delta Y(u))^2\Big),$$

and using conditions **S1-S5** we can further write

$$\begin{aligned}
Y^4(t) \quad = \quad & \int_s^t \Big(4Y^3(u-)\sigma(u-)g(\mathbf{x}) + 6Y^2(u-)\sigma^2(u-)g^2(\mathbf{x}) + 4Y(u-)\sigma(u-)^3 g^3(\mathbf{x}) + \sigma^4(u-)g^4(\mathbf{x})\Big)\tilde{\mu}(du,d\mathbf{x}) \\
& + \int_s^t \sigma^4(u)du \int_{\mathbb{R}_0^n} g^4(\mathbf{x})G(d\mathbf{x}) + 4\int_s^t Y(u)\sigma^3(u)du \int_{\mathbb{R}_0^n} g^3(\mathbf{x})G(d\mathbf{x}) \\
& +6\int_s^t Y^2(u)\sigma^2(u)du \int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}) + Y^4(s).
\end{aligned}$$

Now I show that the first integral is of integrable variation. I demonstrate this with several inequalities. Using the integrability of $\sup\limits_{s<u\le t} Y^4(u)$ and $\sigma^4(s)$, combined with the Hölder's in-

equality we have

$$\mathbb{E}\bigg(\int_s^t Y^3(u-)\sigma(u-)du\bigg) \leq \mathbb{E}\bigg(\sup_{s<u\leq t}|Y^3(u)|\int_s^t \sigma(u-)du\bigg)$$
$$\leq \bigg(\mathbb{E}(\sup_{s<u\leq t} Y^4(u))\bigg)^{3/4}\bigg(\mathbb{E}\bigg(\int_s^t \sigma(u-)du\bigg)^4\bigg)^{1/4}$$
$$\leq (t-s)^{3/4}\bigg(\mathbb{E}(\sup_{s<u\leq t} Y^4(u))\bigg)^{3/4}\bigg(\mathbb{E}\bigg(\int_s^t \sigma^4(u-)du\bigg)\bigg)^{1/4},$$

$$\mathbb{E}\bigg(\int_s^t Y^2(u-)\sigma^2(u-)du\bigg) \leq \mathbb{E}\bigg(\sup_{s<u\leq t}|Y^2(u)|\int_s^t \sigma^2(u-)du\bigg)$$
$$\leq \bigg(\mathbb{E}(\sup_{s<u\leq t} Y^4(u))\bigg)^{1/2}\bigg(\mathbb{E}\bigg(\int_s^t \sigma^2(u-)du\bigg)^2\bigg)^{1/2},$$

$$\mathbb{E}\bigg(\int_s^t Y(u-)\sigma^3(u-)du\bigg) \leq \mathbb{E}\bigg(\sup_{s<u\leq t}|Y(u)|\int_s^t \sigma^3(u-)du\bigg)$$
$$\leq \bigg(\mathbb{E}(\sup_{s<u\leq t} Y^4(u))\bigg)^{1/4}\bigg(\mathbb{E}\bigg(\int_s^t \sigma^3(u-)du\bigg)^{4/3}\bigg)^{3/4}$$
$$\leq (t-s)^{1/3}\bigg(\mathbb{E}(\sup_{s<u\leq t} Y^4(u))\bigg)^{1/4}\bigg(\mathbb{E}\bigg(\int_s^t \sigma^4(u-)du\bigg)\bigg)^{3/4},$$

Using these results, the fact that $\tilde{\mu}$ is a martingale random measure and the condition **S5** we have

$$E(r_a(0)-\mathbb{E}(r_a(0)))^4 = aE(\sigma^4(s))\int_{\mathbb{R}_0^n} g^4(\mathbf{x})G(d\mathbf{x}) + 4a\int_0^a E(Y(s)\sigma^3(s))ds\int_{\mathbb{R}_0^n} g^3(\mathbf{x})G(d\mathbf{x})$$
$$+6\int_0^a E(Y^2(s)\sigma^2(s))ds\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}).$$

For the first term I make use of the result in (A.7). For the second term I use the assumption **S5**. For the last term using again Ito's lemma we have

$$E(Y^2(t)\sigma^2(t)) = E\bigg[\bigg(\int_0^t\int_{\mathbb{R}_0^n}(2Y(s-)\sigma(s-)g(\mathbf{x})+\sigma^2(s-)g^2(\mathbf{x}))\tilde{\mu}(ds,d\mathbf{x}) + \int_0^t \sigma^2(s)ds\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x})\bigg)$$
$$\times\bigg(\int_{-\infty}^t f(t-s)k(\mathbf{x})\tilde{\mu}(ds,d\mathbf{x}) + \int_0^\infty f(s)ds\int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x})\bigg)\bigg].$$

To continue further I make use of the following inequality

$$\mathbb{E}\bigg(\int_0^t Y^2(s-)\sigma^2(s-)ds\bigg) < \infty, \tag{B.2}$$

which was established above. This combined with the square integrability of the state variable

$\sigma^2(t)$ and the properties of the square integrable martingales finally gives

$$
\begin{aligned}
E(Y^2(t)\sigma^2(t)) \quad &= \quad 2\int_0^t E(Y(s)\sigma(s))f(t-s)ds\int_{\mathbb{R}_0^n} g(\mathbf{x})k(\mathbf{x})G(d\mathbf{x}) \\
&\quad +E(\sigma^2(u))\int_0^t f(t-s)ds\int_{\mathbb{R}_0^n} g^2(\mathbf{x})k(\mathbf{x})G(d\mathbf{x}) \\
&\quad +tE(\sigma^2(u))\int_0^\infty f(s)ds\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x})\int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x}) \\
&\quad +\int_{-\infty}^t H^t(0,s)f(t-s)ds\int_{\mathbb{R}_0^n} k^2(\mathbf{x})G(d\mathbf{x})\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}).
\end{aligned}
$$

Using again the assumption in **S5** and combining everything together I have the result for the forth central moment in (30).

I finish by showing the result for the covariance of the returns.

$$\mathrm{Cov}(r_a^2(0), r_a^2(h)) = E(r_a^2(0)r_a^2(h)) - (E(r_a^2(0)))^2$$

Using the fact that $Y(t)$ is defined with respect to the martingale random measure $\tilde{\mu}$ we have

$$
\begin{aligned}
E(r_a^2(0)r_a^2(h)) &= E(Y^a(0)Y^a(h))^2 \\
&= E\Big[Y^2(a)\Big(Y(h+a)-Y(h)\Big)^2\Big] \\
&= E\Big[Y^2(a)\Big(Y^2(h+a)-Y^2(h)\Big)\Big].
\end{aligned}
$$

To simplify further I apply the Ito's formula and condition **S1**

$$
\begin{aligned}
Y^2(t)-Y^2(s) &= \int_s^t 2Y(u-)dY(u) + \sum_{s<u\le t}(\Delta Y(u))^2 \\
&= \int_s^t\int_{\mathbb{R}_0^n} 2Y(u-)\sigma(u-)g(\mathbf{x})\tilde{\mu}(du,d\mathbf{x}) + \int_s^t\int_{\mathbb{R}_0^n}\sigma^2(u-)g^2(\mathbf{x})\mu(du,d\mathbf{x}) \\
&= \int_s^t\int_{\mathbb{R}_0^n}\Big(2Y(u-)\sigma(u-)g(\mathbf{x})+\sigma^2(u-)g^2(\mathbf{x})\Big)\tilde{\mu}(du,d\mathbf{x}) + \int_s^t\sigma^2(u)du\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}).
\end{aligned}
$$

Using this fact I can write

$$
\begin{aligned}
E(r_a^2(0)r_a^2(h)) \quad &= \quad E\Big[\Big(\int_0^a\int_{\mathbb{R}_0^n}\Big(2Y(u-)\sigma(u-)g(\mathbf{x})+\sigma^2(u-)g^2(\mathbf{x})\Big)\tilde{\mu}(du,d\mathbf{x}) + \int_0^a\sigma^2(u)du\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x})\Big) \\
&\quad \times\Big(\int_h^{h+a}\int_{\mathbb{R}_0^n}\Big(2Y(u-)\sigma(u-)g(\mathbf{x})+\sigma^2(u-)g^2(\mathbf{x})\Big)\tilde{\mu}(du,dx) + \int_h^{h+a}\sigma^2(u)du\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x})\Big)\Big] \\
&= \quad E\Big(\int_0^a\sigma^2(s)ds\int_h^{h+a}\sigma^2(s)ds\Big)\Big(\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x})\Big)^2 \\
&\quad +E\Big(\int_0^a\int_{\mathbb{R}_0^n}\Big(2Y(u-)\sigma(u-)g(\mathbf{x})+\sigma^2(u-)g^2(\mathbf{x})\Big)\tilde{\mu}(du,d\mathbf{x})\int_h^{h+a}\sigma^2(u)du\Big)\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}).
\end{aligned}
$$

I simplify each of the two terms separately. For the first term I use the result in (13) and get

$$
\begin{aligned}
E\left(\int_0^a \sigma^2(s)ds \int_h^{h+a} \sigma^2(s)ds\right) &= a^2\left(\int_0^\infty f(s)ds\right)^2 \left(\int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x})\right)^2 \\
&\quad + \int_{-\infty}^a H^a(h,u)H^a(0,u)du \int_{\mathbb{R}_0^n} k^2(\mathbf{x})G(d\mathbf{x}).
\end{aligned}
$$

For the second term we use the representation in (11), the result in (B.2) and the square integrability of $\sigma^2(s)$ to get

$$
\begin{aligned}
& E\left(\int_0^a \int_{\mathbb{R}_0^n} \left(2Y(u-)\sigma(u-)g(\mathbf{x}) + \sigma^2(u-)g^2(\mathbf{x})\right)\tilde{\mu}(du, d\mathbf{x}) \int_h^{h+a} \sigma^2(u)du\right) \\
&= E\left(\int_0^a \int_{\mathbb{R}_0^n} \left(2Y(u-)\sigma(u-)g(\mathbf{x}) + \sigma^2(u-)g^2(\mathbf{x})\right)\tilde{\mu}(du, d\mathbf{x}) \int_{-\infty}^{h+a} \int_{\mathbb{R}_0^n} H^a(h,u)k(\mathbf{x})\mu(du, dx)\right) \\
&= 2\int_0^a E(Y(u)\sigma(u))H^a(h,u)du \int_{\mathbb{R}_0^n} g(\mathbf{x})k(\mathbf{x})G(d\mathbf{x}) + E(\sigma^2(u)) \int_0^a H^a(h,u)du \int_{\mathbb{R}_0^n} g^2(\mathbf{x})k(\mathbf{x})G(d\mathbf{x}).
\end{aligned}
$$

Using the assumption in **S5** we get the result for the covariance of the return process in (31).

□

C Moments of Power Variation Statistics for the JDSV Models with CARMA(2,1) kernel

Theorem 7 *For the jump-diffusion JDSV model given in (2)-(3) with CARMA(2,1) kernel specified in equation (36) assume that the conditions* **H1**-**H3** *and* **H5** *are satisfied and set* $\alpha = 0$. *Then we have*

$$
\mathbb{E}(RV_\delta^M(t)) = \frac{\pi b_0}{\rho_1\rho_2}\int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x}) + \pi \int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}), \tag{C.1}
$$

$$
\begin{aligned}
&\mathbb{E}(FV_\delta^M) = \pi \int_{\mathbb{R}_0^n} g^4(\mathbf{x})G(d\mathbf{x}) + 6M\frac{\delta^2 b_0}{\rho_1\rho_2}\int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x}) \int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}) + \\
&3M\left(\frac{1-e^{\rho_1\delta}}{\rho_1^3}\frac{b_0^2-\rho_1^2}{\rho_2^2-\rho_1^2} + \frac{1-e^{\rho_2\delta}}{\rho_2^3}\frac{b_0^2-\rho_2^2}{\rho_1^2-\rho_2^2} + \frac{\delta b_0^2}{\rho_1^2\rho_2^2}\right)\int_{\mathbb{R}_0^n} k^2(\mathbf{x})G(d\mathbf{x}) + 3M\frac{\delta^2 b_0^2}{\rho_1^2\rho_2^2}\left(\int_{\mathbb{R}_0^n} k(\mathbf{x})G(d\mathbf{x})\right)^2 + \\
&6M\left(\frac{b_0+\rho_1}{\rho_1-\rho_2}\frac{e^{\rho_1\delta}-1}{\rho_1^2} + \frac{b_0+\rho_2}{\rho_2-\rho_1}\frac{e^{\rho_2\delta}-1}{\rho_2^2} + \frac{\delta b_0}{\rho_1\rho_2}\right)\int_{\mathbb{R}_0^n} g^2(\mathbf{x})k(\mathbf{x})G(d\mathbf{x}) + 3M\delta^2\left(\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x})\right)^2,
\end{aligned} \tag{C.2}
$$

$$
\begin{aligned}
&Var(RV_\delta^M(t)) = \\
&2\left(\frac{b_0+\rho_1}{\rho_1-\rho_2}\frac{e^{\rho_1\delta M} - Me^{\rho_1\delta} + M - 1}{\rho_1^2} + \frac{b_0+\rho_2}{\rho_2-\rho_1}\frac{e^{\rho_2\delta M} - Me^{\rho_2\delta} + M - 1}{\rho_2^2}\right)\int_{\mathbb{R}_0^n} g^2(\mathbf{x})k(\mathbf{x})G(d\mathbf{x}) \\
&+ \left(\frac{b_0^2-\rho_1^2}{\rho_1^3(\rho_1^2-\rho_2^2)}(e^{\rho_1\delta M} - Me^{\rho_1\delta} + M - 1) + \frac{b_0^2-\rho_2^2}{\rho_2^3(\rho_2^2-\rho_1^2)}(e^{\rho_2\delta M} - Me^{\rho_2\delta} + M - 1)\right)\int_{\mathbb{R}_0^n} k^2(\mathbf{x})G(d\mathbf{x}) \\
&+ M\mathbb{E}(r_\delta^4(0)) - M\left(\mathbb{E}(r_\delta^2(0))\right)^2.
\end{aligned} \tag{C.3}
$$

For $i = 1, 2...$ *we have*

$$\mathbb{E}(r_{\delta M}(t) RV_\delta^M(t+i))$$
$$= \left(\frac{b_0+\rho_1}{(\rho_1-\rho_2)\rho_1^2} e^{\rho_1(i-\delta M)}(e^{\rho_1 \delta M}-1)^2 + \frac{b_0+\rho_2}{(\rho_2-\rho_1)\rho_2^2} e^{\rho_2(i-\delta M)}(e^{\rho_2 \delta M}-1)^2 \right) \int_{\mathbb{R}_0^n} k(\mathbf{x}) g(\mathbf{x}) G(d\mathbf{x}), \tag{C.4}$$

$$Cov(RV_\delta^M(t), RV_\delta^M(t-i)) = C_1 e^{\rho_1(i-\delta M)} + C_2 e^{\rho_2(i-\delta M)}, \tag{C.5}$$

where

$$C_1 = \frac{b_0+\rho_1}{\rho_1-\rho_2} \frac{(e^{\rho_1\delta M}-1)^2}{\rho_1^2} \int_{\mathbb{R}_0^n} g^2(\mathbf{x}) k(\mathbf{x}) G(d\mathbf{x}) + \frac{(b_0^2-\rho_1^2)(e^{\rho_1\delta M}-1)^2}{2\rho_1^3(\rho_1^2-\rho_2^2)} \int_{\mathbb{R}_0^n} k^2(\mathbf{x}) G(d\mathbf{x}) \tag{C.6}$$

and

$$C_2 = \frac{b_0+\rho_2}{\rho_2-\rho_1} \frac{(e^{\rho_2\delta M}-1)^2}{\rho_2^2} \int_{\mathbb{R}_0^n} g^2(\mathbf{x}) k(\mathbf{x}) G(d\mathbf{x}) + \frac{(b_0^2-\rho_2^2)(e^{\rho_2\delta M}-1)^2}{2\rho_2^3(\rho_2^2-\rho_1^2)} \int_{\mathbb{R}_0^n} k^2(\mathbf{x}) G(d\mathbf{x}). \tag{C.7}$$

Proof: First I compute the following expressions associated with the CARMA(2,1) kernel

$$\int_0^\infty f(s) ds = \frac{b_0}{\rho_1 \rho_2},$$

$$\int_0^a H^a(0,u) du = \int_0^a \int_0^s f(s-u) du ds = \frac{b_0+\rho_1}{\rho_1-\rho_2} \frac{e^{\rho_1 a}-1}{\rho_1^2} + \frac{b_0+\rho_2}{\rho_2-\rho_1} \frac{e^{\rho_2 a}-1}{\rho_2^2} + \frac{a b_0}{\rho_1\rho_2},$$

$$\int_{-\infty}^a (H^a(0,u))^2 du = \frac{1-e^{\rho_1 a}}{\rho_1^3} \frac{b_0^2-\rho_1^2}{\rho_2^2-\rho_1^2} + \frac{1-e^{\rho_2 a}}{\rho_2^3} \frac{b_0^2-\rho_2^2}{\rho_1^2-\rho_2^2} + \frac{a b_0^2}{\rho_1^2 \rho_2^2},$$

$$\int_{-\infty}^a H^a(h,u) H^a(0,u) du = \frac{(1-e^{\rho_1 a})^2 (b_0^2-\rho_1^2)}{2\rho_1^3(\rho_1^2-\rho_2^2)} e^{\rho_1(h-a)} + \frac{(1-e^{\rho_2 a})^2 (b_0^2-\rho_2^2)}{2\rho_2^3(\rho_2^2-\rho_1^2)} e^{\rho_2(h-a)},$$

$$\int_0^a H^a(h,u) du = \frac{b_0+\rho_1}{\rho_1-\rho_2} \left(\frac{1-e^{\rho_1 a}}{\rho_1} \right)^2 e^{\rho_1(h-a)} + \frac{b_0+\rho_2}{\rho_2-\rho_1} \left(\frac{1-e^{\rho_2 a}}{\rho_2} \right)^2 e^{\rho_2(h-a)},$$

$$\int_{-\infty}^a H^a(h,u) du = \frac{b_0+\rho_1}{\rho_2-\rho_1} \frac{e^{\rho_1 a}-1}{\rho_1^2} e^{\rho_1(h-a)} + \frac{b_0+\rho_2}{\rho_1-\rho_2} \frac{e^{\rho_2 a}-1}{\rho_2^2} e^{\rho_2(h-a)}.$$

The expression for the mean of the Realized Variance and FV follows from

$$\mathbb{E}(RV_\delta^M(t)) = M\mathbb{E}(r_\delta^2(s)) \qquad \mathbb{E}(FV_\delta^M(t)) = M\mathbb{E}(r_\delta^4(s)),$$

and the expressions for the mean, variance and forth central moments of the return process given in equations (16)-(19) respectively.

For the covariance and the variance of the Realized Variance I first compute the covariance between the squared demeaned high frequency returns using the quantities for the CARMA(2,1) kernel calculated above and Theorem 3

$$\mathrm{Cov}(r_a^2(0), r_a^2(h)) =$$
$$\left(\frac{b_0+\rho_1}{\rho_1-\rho_2}\left(\frac{1-e^{\rho_1 a}}{\rho_1}\right)^2 e^{\rho_1(h-a)} + \frac{b_0+\rho_2}{\rho_2-\rho_1}\left(\frac{1-e^{\rho_2 a}}{\rho_2}\right)^2 e^{\rho_2(h-a)}\right)\int_{\mathbb{R}_0^n} g^2(\mathbf{x})k(\mathbf{x})G(d\mathbf{x})$$
$$+\left(\frac{(1-e^{\rho_1 a})^2(b_0^2-\rho_1^2)}{2\rho_1^3(\rho_1^2-\rho_2^2)}e^{\rho_1(h-a)} + \frac{(1-e^{\rho_2 a})^2(b_0^2-\rho_2^2)}{2\rho_2^3(\rho_2^2-\rho_1^2)}e^{\rho_2(h-a)}\right)\int_{\mathbb{R}_0^n} k^2(\mathbf{x})G(d\mathbf{x}).$$

Then I use the following expression for the variance and covariance and the Realized Variance

$$\mathrm{Cov}(RV_\delta^M(t), RV_\delta^M(t-i)) = \sum_{k=-(M-1)}^{M-1}(M-|k|)\mathrm{Cov}(r_\delta^2(0), r_\delta^2(i-k\delta)), \qquad \text{for } i = 0, 1, ... \tag{C.8}$$

To simplify further the expressions for the covariance between RV and past intraday returns, the covariance and variance of the RV I make use of the following expressions for arbitrary ρ

$$\sum_{k=-(M-1)}^{M-1}(M-|k|)e^{\rho(i-\delta k)} = M\sum_{k=-(M-1)}^{M-1} e^{\rho(i-\delta k)} - \left(\sum_{k=1}^{M-1} k e^{\rho(i-\delta k)} + \sum_{k=1}^{M-1} k e^{\rho(i+\delta k)}\right),$$

and

$$\sum_{k=1}^{M-1} k e^{\rho\delta k} = \frac{(M-1)e^{(M+1)\rho\delta} - Me^{M\rho\delta} + e^{\rho\delta}}{(e^{\rho\delta}-1)^2}. \tag{C.9}$$

Therefore

$$\sum_{k=-(M-1)}^{M-1}(M-|k|)e^{\rho(i-\delta k)} = e^{\rho\delta}\frac{(e^{\rho\delta M}-1)^2}{(e^{\rho\delta}-1)^2}e^{\rho(i-\delta M)}. \tag{C.10}$$

□

Theorem 8 *For the pure jump JDSV model in (24)-(25) with CARMA(2,1) kernel specified in (36), assume that the conditions* **S1**-**S5** *are satisfied. Then if $\alpha = 0$ we have*

$$\mathbb{E}(RV_\delta^M(t)) = \frac{\pi b_0}{\rho_1\rho_2}\int_{R_0^n} k(x)G(dx)\int_{R_0^n} g^2(x)G(dx), \tag{C.11}$$

$$\begin{aligned}
E(FV_\delta^M(t)) \quad = \quad & \pi\left(\frac{b_0^2-\rho_1^2}{2\rho_1(\rho_1^2-\rho_2^2)} + \frac{b_0^2-\rho_2^2}{2\rho_2(\rho_2^2-\rho_1^2)}\right)\int_{R_0^n} k^2(x)G(dx)\int_{R_0^n} g^4(x)G(dx) \\
& + \frac{\pi b_0^2}{\rho_1^2\rho_2^2}\left(\int_{R_0^n} k(x)G(dx)\right)^2\int_{R_0^n} g^4(x)G(dx) \\
& + 6M\frac{b_0}{\rho_1\rho_2}\frac{b_0+\rho_1}{\rho_1-\rho_2}\frac{(e^{\delta\rho_1}-1-\delta\rho_1)}{\rho_1^2}\int_{R_0^n} g^2(x)G(dx)\int_{R_0^n} k(x)G(dx)\int_{R_0^n} g^2(x)k(x)G(dx) \\
& + 6M\frac{b_0}{\rho_1\rho_2}\frac{b_0+\rho_2}{\rho_2-\rho_1}\frac{(e^{\delta\rho_2}-1-\delta\rho_2)}{\rho_2^2}\int_{R_0^n} g^2(x)G(dx)\int_{R_0^n} k(x)G(dx)\int_{R_0^n} g^2(x)k(x)G(dx) \\
& + 3\frac{\pi\delta b_0^2}{\rho_1^2\rho_2^2}\left(\int_{R_0^n} g^2(x)G(dx)\int_{R_0^n} k(x)G(dx)\right)^2 \\
& + 6M\frac{b_0^2-\rho_1^2}{\rho_2^2-\rho_1^2}\frac{\delta\rho_1+1-e^{\delta\rho_1}}{2\rho_1^3}\int_{R_0^n} k^2(x)G(dx)\left(\int_{R_0^n} g^2(x)G(dx)\right)^2 \\
& + 6M\frac{b_0^2-\rho_2^2}{\rho_1^2-\rho_2^2}\frac{\delta\rho_2+1-e^{\delta\rho_2}}{2\rho_2^3}\int_{R_0^n} k^2(x)G(dx)\left(\int_{R_0^n} g^2(x)G(dx)\right)^2,
\end{aligned} \tag{C.12}$$

$$
\begin{aligned}
Var(RV_\delta^M(t)) =& 2\frac{b_0}{\rho_1\rho_2}\frac{b_0+\rho_1}{\rho_1-\rho_2}\frac{e^{\rho_1\delta M}-Me^{\rho_1\delta}+M-1}{\rho_1^2}\int_{R_0^n} g^2(x)G(dx)\int_{R_0^n} k(x)G(dx)\int_{R_0^n} g^2(x)k(x)G(dx)\\
&+2\frac{b_0}{\rho_1\rho_2}\frac{b_0+\rho_2}{\rho_2-\rho_1}\frac{e^{\rho_2\delta M}-Me^{\rho_2\delta}+M-1}{\rho_2^2}\int_{R_0^n} g^2(x)G(dx)\int_{R_0^n} k(x)G(dx)\int_{R_0^n} g^2(x)k(x)G(dx)\\
&+\frac{b_0^2-\rho_1^2}{\rho_1^3(\rho_1^2-\rho_2^2)}(e^{\rho_1\delta M}-Me^{\rho_1\delta}+M-1)\int_{R_0^n} k^2(x)G(dx)\bigg(\int_{R_0^n} g^2(x)G(dx)\bigg)^2\\
&+\frac{b_0^2-\rho_2^2}{\rho_2^3(\rho_2^2-\rho_1^2)}(e^{\rho_2\delta M}-Me^{\rho_2\delta}+M-1)\int_{R_0^n} k^2(x)G(dx)\bigg(\int_{R_0^n} g^2(x)G(dx)\bigg)^2\\
&+ME(r_\delta^4(0)-M\bigg(E(r_\delta^2(0))\bigg)^2,
\end{aligned}
\tag{C.13}
$$

and for $i = 1, 2...$

$$
Cov(RV_\delta^M(t), RV_\delta^M(t-i)) = C_1 e^{\rho_1(i-\delta M)} + C_2 e^{\rho_2(i-\delta M)}, \tag{C.14}
$$

where

$$
\begin{aligned}
C1 =& \frac{b_0}{\rho_1\rho_2}\frac{b_0+\rho_1}{\rho_1-\rho_2}\frac{(e^{\rho_1\delta M}-1)^2}{\rho_1^2}\int_{R_0^n} g^2(x)G(dx)\int_{R_0^n} k(x)G(dx)\int_{R_0^n} g^2(x)k(x)G(dx)\\
&+\frac{(b_0^2-\rho_1^2)(e^{\rho_1\delta M}-1)^2}{2\rho_1^3(\rho_1^2-\rho_2^2)}\int_{R_0^n} k^2(x)G(dx)\bigg(\int_{R_0^n} g^2(x)G(dx)\bigg)^2,
\end{aligned}
$$

and

$$
\begin{aligned}
C2 =& \frac{b_0}{\rho_1\rho_2}\frac{b_0+\rho_2}{\rho_2-\rho_1}\frac{(e^{\rho_2\delta M}-1)^2}{\rho_2^2}\int_{R_0^n} g^2(x)G(dx)\int_{R_0^n} k(x)G(dx)\int_{R_0^n} g^2(x)k(x)G(dx)\\
&+\frac{(b_0^2-\rho_2^2)(e^{\rho_2\delta M}-1)^2}{2\rho_2^3(\rho_2^2-\rho_1^2)}\int_{R_0^n} k^2(x)G(dx)\bigg(\int_{R_0^n} g^2(x)G(dx)\bigg)^2.
\end{aligned}
$$

Proof: The theorem is a direct consequence of the result in Theorem 6, the definitions of realized variance and forth variation in equations (33) and (34), the results used in the proof of Theorem 7, combined with the following additional properties associated with the CARMA(2,1) kernel.

$$
\begin{aligned}
cov(r_a^2(0), r_a^2(h)) \quad = \quad & \bigg[\frac{(1-e^{\rho_1 a})^2(b_0^2-\rho_1^2)}{2\rho_1^3(\rho_1^2-\rho_2^2)}e^{\rho_1(h-a)} + \frac{(1-e^{\rho_2 a})^2(b_0^2-\rho_2^2)}{2\rho_2^3(\rho_2^2-\rho_1^2)}e^{\rho_2(h-a)}\bigg]\\
&\times\int_{R_0^n} k^2(x)G(dx)\bigg(\int_{R_0^n} g^2(x)G(dx)\bigg)^2\\
&+\bigg[\bigg(\frac{b_0+\rho_1}{\rho_1-\rho_2}\bigg)\bigg(\frac{1-e^{\rho_1 a}}{\rho_1}\bigg)^2 e^{\rho_1(h-a)} + \bigg(\frac{b_0+\rho_2}{\rho_2-\rho_1}\bigg)\bigg(\frac{1-e^{\rho_2 a}}{\rho_2}\bigg)^2 e^{\rho_2(h-a)}\bigg]\\
&\times\frac{b_0}{\rho_1\rho_2}\int_{R_0^n} g^2(x)G(dx)\int_{R_0^n} k(x)G(dx)\int_{R_0^n} g^2(x)k(x)G(dx),
\end{aligned}
\tag{C.15}
$$

$$
\int_0^\infty f^2(s)ds = \frac{b_0^2-\rho_1^2}{2\rho_1(\rho_1^2-\rho_2^2)} + \frac{b_0^2-\rho_2^2}{2\rho_2(\rho_2^2-\rho_1^2)},
$$

$$\int_{-\infty}^{u} H^u(0,s)f(u-s)ds = \frac{b_0^2-\rho_1^2}{\rho_2^2-\rho_1^2}\frac{1-e^{\rho_1 u}}{2\rho_1^2} + \frac{b_0^2-\rho_2^2}{\rho_1^2-\rho_2^2}\frac{1-e^{\rho_2 u}}{2\rho_2^2}.$$

□

D Moments in the Affine Jump Diffusion Model

Theorem 9 *In the affine jump diffusion model (43)-(46), assume that $W(t)$, $B_1(t)$ and $B_2(t)$ are independent Brownian motions. Further assume that the parameters in (45)-(46) satisfy*

$$\kappa_i > 0, \qquad \theta_i > 0 \qquad \textit{and} \qquad \sigma_i^2 \leq 2\kappa_i\theta_i, \qquad \textit{for } i=1,2.$$

Then for $\alpha = 0$

$$\mathbb{E}(RV_\delta^M(t)) = \pi(\theta_1+\theta_2) + \pi\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}), \tag{D.1}$$

$$\begin{aligned} Cov(RV_\delta^M(t), RV_\delta^M(t-i)) &= e^{-\kappa_1(i-\delta M)}\left(\frac{1-e^{-\kappa_1\delta M}}{\kappa_1}\right)^2\frac{\sigma_1^2\theta_1}{2\kappa_1} \\ &\quad + e^{-\kappa_2(i-\delta M)}\left(\frac{1-e^{-\kappa_2\delta M}}{\kappa_2}\right)^2\frac{\sigma_2^2\theta_2}{2\kappa_2}, \end{aligned} \tag{D.2}$$

$$\begin{aligned} Var(RV_\delta^M(t)) &= \frac{e^{-\kappa_1\delta M}-Me^{-\kappa_1\delta}+M-1}{\kappa_1^2}\frac{\sigma_1^2\theta_1}{\kappa_1} + \frac{e^{-\kappa_2\delta M}-Me^{-\kappa_2\delta}+M-1}{\kappa_2^2}\frac{\sigma_2^2\theta_2}{\kappa_2} \\ &\quad + M\mathbb{E}(r_\delta^4(0)) - M\left(\mathbb{E}(r_\delta^2(0))\right)^2, \end{aligned} \tag{D.3}$$

$$\begin{aligned} \mathbb{E}(FV_\delta^M(t)) &= 3M\delta^2(\theta_1+\theta_2)^2 + 6M\delta^2(\theta_1+\theta_2)\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}) \\ &\quad + M\delta\int_{\mathbb{R}_0^n} g^4(\mathbf{x})G(d\mathbf{x}) + 3M\delta^2\left(\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x})\right)^2 \\ &\quad + 3M\frac{\sigma_1^2\theta_1}{\kappa_1^3}(\delta\kappa_1 - 1 + e^{-\kappa_1\delta}) + 3M\frac{\sigma_2^2\theta_2}{\kappa_2^3}(\delta\kappa_2 - 1 + e^{-\kappa_2\delta}). \end{aligned} \tag{D.4}$$

Proof: By an application of Ito's theorem for i=1,2 we have

$$de^{\kappa_i t}V_i(t) = e^{\kappa_i t}\kappa_i\theta_i dt + \sigma_i e^{\kappa_i t}\sqrt{V_i(t)}dB_i(t),$$

therefore for $t \geq s$

$$\mathbb{E}(V_i(t)|\mathcal{F}_s) = e^{-\kappa_i(t-s)}V_i(s) + \theta_i(1-e^{-\kappa_i(t-s)}). \tag{D.5}$$

Another application of Ito's theorem gives for $i=1,2$

$$de^{2\kappa_i t}V_i^2(t) = e^{2\kappa_i t}(2\kappa_i\theta_i + \sigma_i^2)V_i(t)dt + 2\sigma_i e^{2\kappa_i t}V_i(t)\sqrt{V_i(t)}dB_i(t),$$

and therefore for $t \geq s$

$$\mathbb{E}(V_i^2(t)|\mathcal{F}_s) = e^{-2\kappa_i(t-s)}V_i^2(s) + (2\kappa_i\theta_i + \sigma_i^2)\int_s^t e^{2\kappa_i(u-t)}\mathbb{E}(V_i(u)|\mathcal{F}_s)du,$$

and from here

$$\begin{aligned}\mathbb{E}(V_i^2(t)|\mathcal{F}_s) &= e^{-2\kappa_i(t-s)}V_i^2(s) + \left(\frac{2\kappa_i\theta_i + \sigma_i^2}{\kappa_i}\right)(e^{-\kappa_i(t-s)} - e^{-2\kappa_i(t-s)})V_i(t) \\ &\quad + \left(\frac{(2\kappa_i\theta_i + \sigma_i^2)\theta_i}{2\kappa_i}\right)(1 - e^{-\kappa_i(t-s)})^2. \end{aligned} \tag{D.6}$$

The first two unconditional moments of $V_i(t)$, provided it is covariance stationary process (which is the case under the conditions on the parameters in the theorem) are

$$\mathbb{E}(V_i(t)) = \theta_i \qquad \text{and} \qquad \mathbb{E}(V_i^2(t)) = \frac{(2\kappa_i\theta_i + \sigma_i^2)\theta_i}{2\kappa_i}.$$

For further use we also derive the following

$$\begin{aligned}\mathbb{E}(V_i(u)V_i(s)) &= \mathbb{E}(V_i(u)\mathbb{E}(V_i(s)|\mathcal{F}_u)) \\ &= e^{-\kappa_i(s-u)}\frac{\sigma_i^2\theta_i}{2\kappa_i} + \theta_i^2, \qquad \text{for } s \geq u,\ i = 1, 2.\end{aligned} \tag{D.7}$$

The mean of the squared returns over a period with length a is

$$\begin{aligned}\mathbb{E}(r_a^2(t)) &= \mathbb{E}\left(\int_t^{t+a} V_i(s)ds\right) + \mathbb{E}\left(\int_t^{t+a}\int_{\mathbb{R}_0^n} g(\mathbf{x})\tilde{\mu}(ds, d\mathbf{x})\right)^2 \\ &= a(\theta_1 + \theta_2) + a\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}).\end{aligned}$$

From here and the definition of realized variance, follows the expression in (D.1).

For the forth power of the return over period with length a, we have

$$\begin{aligned}\mathbb{E}(r_a^4(t)) &= \mathbb{E}\left(\int_t^{t+a}\sqrt{V(t)}dW(t)\right)^4 + 6\mathbb{E}\left(\int_t^{t+a}\sqrt{V(t)}dW(t)\int_t^{t+a}\int_{\mathbb{R}_0^n} g(\mathbf{x})\tilde{\mu}(ds, d\mathbf{x})\right)^2 \\ &\quad + \mathbb{E}\left(\int_t^{t+a}\int_{\mathbb{R}_0^n} g(\mathbf{x})\tilde{\mu}(ds, d\mathbf{x})\right)^4 \\ &= 3\mathbb{E}\left(\left(\int_t^{t+a} V(s)ds\right)^2\right) + 6\mathbb{E}\left(\int_t^{t+a} V(s)ds\right)\mathbb{E}\left(\int_t^{t+a}\int_{\mathbb{R}_0^n} g(\mathbf{x})\tilde{\mu}(ds, d\mathbf{x})\right)^2 \\ &\quad + a\int_{\mathbb{R}_0^n} g^4(\mathbf{x})G(d\mathbf{x}) + 3a^2\left(\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x})\right)^2 \\ &= 3\mathbb{E}\left(\left(\int_t^{t+a} V(s)ds\right)^2\right) + 6a^2(\theta_1 + \theta_2)\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}) \\ &\quad + a\int_{\mathbb{R}_0^n} g^4(\mathbf{x})G(d\mathbf{x}) + 3a^2\left(\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x})\right)^2,\end{aligned} \tag{D.8}$$

where I made use of the fact that conditional on the volatility the continuous component of the return process is Gaussian.

To continue further I derive the mean of the squared integrated variance. We use the independence and stationarity of the two volatility factors as well as the result in (D.7) to get

$$\begin{aligned}
\mathbb{E}\Big(\Big(\int_t^{t+a} V(s)ds\Big)^2\Big) &= \int_0^a \int_0^a \mathbb{E}(V(s)V(u)))dsdu \\
&= 2\int_0^a \int_u^a \mathbb{E}(V_1(s)V_1(u)))dsdu + 2\int_0^a \int_u^a \mathbb{E}(V_2(s)V_2(u)))dsdu + 2a^2\theta_1\theta_2 \\
&= a^2(\theta_1+\theta_2)^2 + \frac{\sigma_1^2\theta_1}{\kappa_1^3}(e^{\kappa_1 a} + a\kappa_1 - 1) + \frac{\sigma_2^2\theta_2}{\kappa_2^3}(e^{\kappa_2 a} + a\kappa_2 - 1).
\end{aligned}$$

From here and the definition of the forth variation its mean is easily deduced.

Next I derive the covariance of the squared returns over period with length a, which is in turn used for deriving the covariance and the variance of the realized variance. For $h = a, 2a, ...$

$$Cov(r_a^2(t), r_a^2(t+h)) = \mathbb{E}(r_a^2(t)r_a^2(t+h)) - (\mathbb{E}(r_a^2(t)))^2.$$

$$\begin{aligned}
\mathbb{E}(r_a^2(t)r_a^2(t+h)) &= \mathbb{E}\Big[\Big(\int_t^{t+a} \sqrt{V(s)}dW(s) + \int_t^{t+a}\int_{\mathbb{R}_0^n} g(\mathbf{x})\tilde{\mu}(ds,d\mathbf{x})\Big)^2 \\
&\quad \times\Big(\int_{t+h}^{t+h+a} \sqrt{V(s)}dW(s) + \int_{t+h}^{t+h+a}\int_{\mathbb{R}_0^n} g(\mathbf{x})\tilde{\mu}(ds,d\mathbf{x})\Big)^2\Big] \\
&= \mathbb{E}\Big(\int_t^{t+a} V(s)ds \int_{t+h}^{t+h+a} V(s)ds\Big) + 2a^2(\theta_1+\theta_2)\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x}) \\
&\quad + a^2\Big(\int_{\mathbb{R}_0^n} g^2(\mathbf{x})G(d\mathbf{x})\Big)^2,
\end{aligned} \tag{D.9}$$

where I made use of the independence of the continuous and the jump component of the returns, as well as the time homogeneity of the Poisson random measure μ.

Therefore

$$\begin{aligned}
Cov(r_a^2(t), r_a^2(t+h)) &= \mathbb{E}\Big(\int_t^{t+a} V(s)ds \int_{t+h}^{t+h+a} V(s)ds\Big) - (\theta_1+\theta_2)^2 \\
&= \mathbb{E}\Big(\int_t^{t+a} V_1(s)ds \int_{t+h}^{t+h+a} V_1(s)ds\Big) - a^2\theta_1^2 \\
&\quad + \mathbb{E}\Big(\int_t^{t+a} V_2(s)ds \int_{t+h}^{t+h+a} V_2(s)ds\Big) - a^2\theta_2^2.
\end{aligned} \tag{D.10}$$

Thus we need the covariance of the integrated variance. By the law of the iterated expectation we have

$$\mathbb{E}\Big(\int_t^{t+a} V_i(s)ds \int_{t+h}^{t+h+a} V_i(s)ds\Big) = \mathbb{E}\Big(\int_t^{t+a} V_i(s)ds\,\mathbb{E}\Big(\int_{t+h}^{t+h+a} V_i(s)ds\Big|\mathcal{F}_{t+a}\Big)\Big),$$

and using (D.5)

$$\mathbb{E}\bigg(\int_{t+h}^{t+h+a} V_i(s)ds\bigg|\mathcal{F}_{t+a}\bigg) = \frac{V_i(t+a)}{\kappa_i}(e^{-\kappa_i(h-a)} - e^{-\kappa_i h}) + a\theta_i - \theta_i\frac{e^{-\kappa_i(h-a)} - e^{-\kappa_i h}}{\kappa_i}.$$

Therefore

$$\begin{aligned}\mathbb{E}\bigg(\int_{t}^{t+a} V_i(s)ds\int_{t+h}^{t+h+a} V_i(s)ds\bigg) &= \mathbb{E}\bigg(\int_{t}^{t+a} V_i(s)V_i(t+a)ds\bigg)\bigg(\frac{e^{-\kappa_i(h-a)} - e^{-\kappa_i h}}{\kappa_i}\bigg)\\ &\quad +a\theta_i\bigg(a\theta_i - \theta_i\frac{e^{-\kappa_i(h-a)} - e^{-\kappa_i h}}{\kappa_i}\bigg).\end{aligned} \tag{D.11}$$

Using the result in (D.7) we have

$$\mathbb{E}\bigg(\int_{t}^{t+a} V_i(s)ds\int_{t+h}^{t+h+a} V_i(s)ds\bigg) = \bigg(\frac{e^{-\kappa_i(h-a)} - e^{-\kappa_i h}}{\kappa_i}\bigg)\bigg(\frac{1-e^{-\kappa_i a}}{\kappa_i}\bigg)\frac{\sigma_i^2\theta_i}{2\kappa_i} + a^2\theta_i^2.$$

From here follows

$$Cov(r_a^2(t), r_a^2(t+h)) = e^{-\kappa_1(h-a)}\bigg(\frac{1-e^{-\kappa_1 a}}{\kappa_1}\bigg)^2\frac{\sigma_1^2\theta_1}{2\kappa_1} + e^{-\kappa_2(h-a)}\bigg(\frac{1-e^{-\kappa_2 a}}{\kappa_2}\bigg)^2\frac{\sigma_2^2\theta_2}{2\kappa_2}.$$

Then using the expression for the variance and the covariance in equation (C.8), as well as (C.9)-(C.10) we deduce the expressions in the theorem for the variance and the covariance of the realized variance.

□

E Minimum Distance Estimator

Denote with θ $k \times 1$ parameter vector and with $g_T(\theta)$ a $p \times 1$ vector of parameters and the data ($p \geq q$). Then the estimator used here could be written as

$$\hat{\theta} = \frac{1}{2}argmin_{\theta\in\Theta} g_T(\theta)'\hat{W}g_T(\theta), \tag{E.1}$$

where $\hat{W} \xrightarrow{p} W$ is a $q \times q$ positive definite weighting matrix. This is a minimum distance estimator (see Newey and McFadden (1994)).

Under standard conditions (see Newey and McFadden (1994)) the estimator will be consistent and asymptotically normal. The efficient weighting matrix is

$$W_{eff} = (Avar(\sqrt{T}g_T(\theta_0))))^{-1}. \tag{E.2}$$

In the case I am working with

$$g_T(\theta) = P\bigg(\frac{1}{T}\sum_{t=1}^{T} z_t\bigg) - P(m(\theta)), \tag{E.3}$$

where $P(\cdot) : \mathbb{R}^n \to \mathbb{R}^q$, z is a $n \times 1$ data vector and $m(\theta)$ is a $n \times 1$ vector of the parameters such that $m(\theta_0) = \mathbb{E}(z_t)$.

Assuming that $\{z_t\}$ is a covariance stationary process such that the Central Limit Theorem holds (see Wooldridge (1994) for example) we have

$$\sqrt{T}\left(\frac{1}{T}\sum_{t=1}^{T} z_t - m(\theta_0)\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, V_m),$$

where

$$V_m = \Xi_0 + \sum_{i=1}^{\infty}(\Xi_i + \Xi_i')$$

and $\Xi_i = \mathbb{E}(z_t z_{t-i}')$.

Assuming $P(\cdot)$ is differentiable on the range of values $\frac{1}{T}\sum_{t=1}^{T} z_t$ takes and using the Delta method we have

$$\sqrt{T} g_T(\theta_0) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \nabla_m P(m(\theta_0)) V_m \nabla_m P(m(\theta_0)')).$$

Therefore a consistent estimator for the efficient weighting matrix W_{eff} is given by

$$\hat{W}_n = \left(\nabla_m P\left(\frac{1}{T}\sum_{t=1}^{T} z_t\right)\left(\hat{\Xi}_0 + \sum_{i=1}^{L_T}\omega(i,T)(\hat{\Xi}_i + \hat{\Xi}_i')\right)\nabla_m P\left(\frac{1}{T}\sum_{t=1}^{T} z_t\right)'\right)^{-1}, \tag{E.4}$$

where L_T is the number of lags used in estimating the long run variance of the moment vector and

$$\hat{\Xi}_i = \frac{1}{T}\sum_{t=i+1}^{T} z_t z_{t-i}'.$$

For $\omega(i,T)$ I use Bartlett weights (Newey and West (1987))

$$\omega(i,T) = \begin{cases} 1 - \frac{i}{L_T+1} & \text{for } i \le L_T \\ 0 & \text{for } i > L_T. \end{cases} \tag{E.5}$$

Table 1: Summary Statistics for the $DM/\$$ exchange rate data

	daily returns	RV	FV
mean	0.0026	0.5084	0.0369
variance	0.4624	0.2051	0.1401
skewness	-0.0232	3.9249	34.112
kurtosis	5.1615	26.8766	1426.6
min.	-3.2487	0.0518	0.00008
max.	3.5271	5.2454	16.9304

Note: The data spans the period from January 1 1986 till June 30 1999, for a total of 3045 daily observations. The power variation statistics (RV and FV) were computed using 288 five-minute intraday returns and were defined in equations (33) and (34) respectively.

Table 2: Parameter Estimates for the Jump-Diffusion JDSV Model

	CARMA(1,0) no price jumps	CARMA(1,0) with price jumps	CARMA(2,1) no price jumps	CARMA(2,1) with price jumps
β_0			0.1214 (0.0093)	0.16376 (0.0119)
h_1	1.2884 (0.0951)	1.2495 (0.0696)	14.38 (0.1236)	27.791 (0.1158)
h_2			0.91522 (0.0181)	0.46979 (0.0147)
m_c	0.51154 (0.0156)	0.36969 (0.0109)	0.51703 (0.0122)	0.45526 (0.0113)
m_d		0.1109 (0.008)		0.044617 (0.0036)
v	0.21478 (0.0211)	0.11627 (0.0100)	0.24359 (0.0146)	0.21161 (0.0185)
J test of overidentifying restrictions				
χ^2	99.216	50.596	65.994	7.7718
d.o.f	(9)	(8)	(7)	(6)
p-value	0.000	0.000	0.000	0.2553

Note: The table reports the parameter estimates for the stochastic volatility model given in (2)-(3) with the two choices for the memory function $f(\cdot)$: CARMA(1,0) and CARMA(2,1) and jump specification given in assumption **H6**, applied to the $DM/\$$ exchange rate. $h_i = log(0.5)/\rho_i$ for $i = 1, 2$ is the half life of the autoregressive parameters. The data spans the period from December 1 1986 till June 30 1999, for a total of 3045 daily observations, each of which consists of 288 five-minute returns. The power variation statistics were computed using the intraday five-minute returns. The asymptotic variance-covariance matrix is calculated using Bartlett weights with a lag-length of eighty.

Table 3: Parameter Estimates for the Pure Jump JDSV Model

	CARMA(1,0)	CARMA(2,1)
β_0		0.0089722 (0.0004)
h_1	1.83 (0.0478)	76.751 (0.2265)
h_2		0.23969 (0.0169)
m	0.37872 (0.0095)	0.54254 (0.0174)
v	0.098816 (0.0014)	0.34137 (0.0197)
J test of overidentifying restrictions		
χ^2	323.86	135.362
d.o.f	(9)	(7)
p-value	0.000	0.000

Note: The table reports the parameter estimates for the stochastic volatility model given in (24)-(25) with the two choices for the memory function $f(\cdot)$: CARMA(1,0) and CARMA(2,1) and jump specification given in **S6** and **S7**, applied to the $DM/\$$ exchange rate. $h_i = log(0.5)/\rho_i$ for $i = 1, 2$ is the half life of the autoregressive parameters. The data spans the period from December 1 1986 till June 30 1999, for a total of 3045 daily observations, each of which consists of 288 five-minute returns. The power variation statistics were computed using the intraday five-minute returns. The asymptotic variance-covariance matrix is calculated using Bartlett weights with a lag-length of eighty.

Table 4: Parameter Estimates for the Affine Jump Diffusion SV Model

	one factor no price jumps	one factor with price jumps	two factor no price jumps	two factor with price jumps
h_1	1.2883 (0.0908)	1.2509 (0.0855)	22.718 (0.2423)	31.264 (0.2081)
θ_1	0.51154 (0.0156)	0.46844 (0.0191)	0.13129 (0.0190)	0.15985 (0.0177)
σ_1	0.64801 (0.0264)	0.6297 (0.0267)	0.089417 (0.0117)	0.084167 (0.0107)
h_2			1.1463 (0.0298)	0.86017 (0.0256)
θ_2			0.42548 (0.0128)	0.38409 (0.0126)
σ_2			0.71722 (0.0126)	0.78668 (0.0134)
m		0.011536 (0.0176)		0.000037 (0.0063)
v		0.034119 (0.0048)		0.037048 (0.0046)
J test of overidentifying restrictions				
χ^2	99.044	50.596	88.904	37.792
d.o.f	(9)	(7)	(6)	(4)
p-value	0.000	0.000	0.000	0.000

Note: The table reports the parameter estimates for the affine jump diffusion stochastic volatility model given in (43)-(46), applied to the $DM/\$$ exchange rate. $h_i = log(2)/\kappa_i$ for $i = 1, 2$ is the half life of the autoregressive parameters. The data spans the period from December 1 1986 till June 30 1999, for a total of 3045 daily observations, each of which consists of 288 five-minute returns. The power variation statistics were computed using the intraday five-minute returns. The asymptotic variance-covariance matrix is calculated using Bartlett weights with a lag-length of eighty.

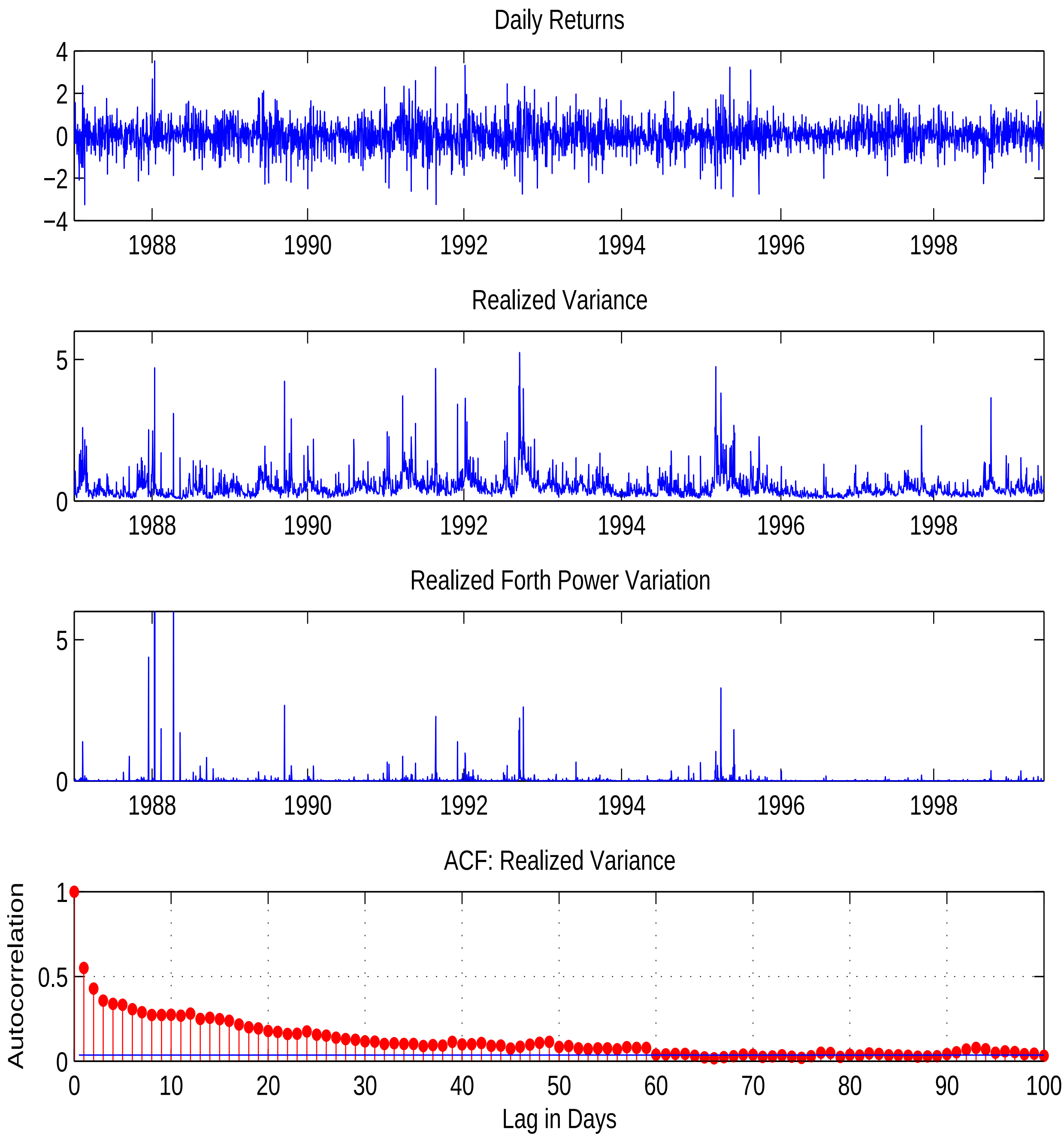

Figure 1: Sumary Statistics for the $DM/\$$ exchange rate data. Top Panel shows the daily returns. Second Panel displays the Realized Variance computed using 288 five-minute intraday returns and defined in equation (33). The third panel shows the Realized Forth Power Variation computed using 288 five-minute intraday returns and defined in equation (34). The last panel shows the autocorrelation in the Realized Variance.

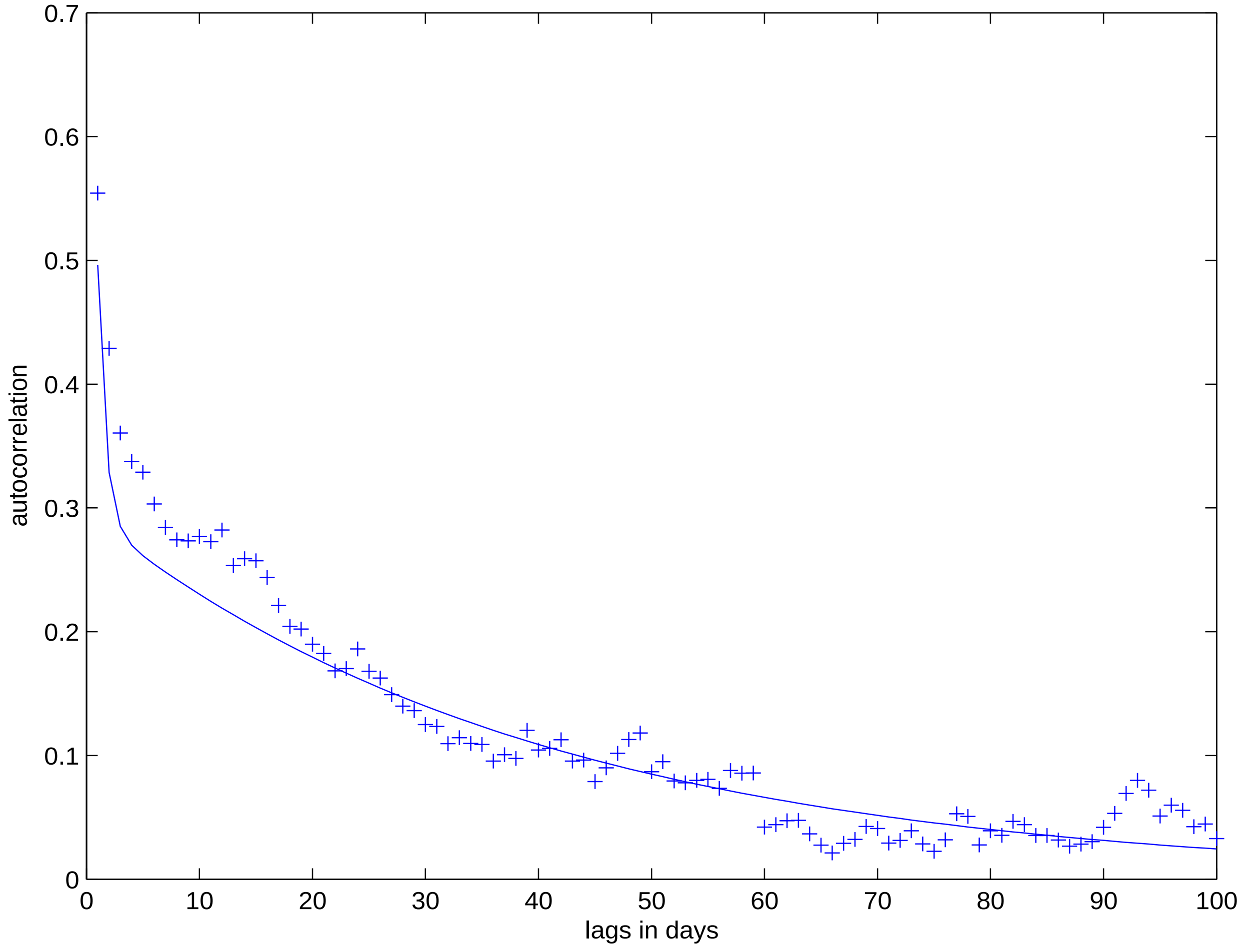

Figure 2: The figure shows the fit of the CARMA(2,1) kernel. The empirical autocorrelation of the Realized Variance is marked with $+$. The solid line is the autocorrelation implied from the jump-diffusion JDSV model given in (2)-(3) with CARMA(2,1) memory function and jump specification given in assumption **H6**. The parameters were set at the estimated values reported in the last column of Table 2.

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