

SAMSI Credit Risk Working Group Presentation

Modeling Correlated Defaults: First Passage Model under Stochastic Volatility

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(Ongoing Project)

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Outline of This Talk

- Brief review of first passage model for single-name default
- Stochastic volatility effects on first passage model for single-name default
- Setup of first passage model under stochastic volatility for two-name correlated defaults
- Derivation for the **approximate** joint default/survival probability
- Model calibration (ongoing work)

First Passage Model for Single-Name Default

- Was first proposed by Black and Cox (1976), where the volatility for the underlying firm's value is taken to be constant.
- The firm defaults at the first time that the firm's value drops to or below some exogenously prespecified level — the default barrier/threshold.
- Closed-form formula exists for the default probability.
- The default probability and yield spread of the bond are close to zero for **short maturities**, which is contrary to market observations.

Stochastic Volatility Effects on Single-Name Default

- Proposed by Fouque, Sircar and Solna (2004).
- No closed-form formula exists for the default probability.
- For fast and slow mean-reverting stochastic volatilities, closed-form formulas exist for the **asymptotic** default probability.
- Incorporating **fast** mean-reverting stochastic volatility can significantly increase the default probability and yield spread for **short maturities**.
- Incorporating **slow** mean-reverting stochastic volatility can increase the default probability and yield spread for **long maturities**.

Model Setup for Two-Name Correlated Defaults

Under physical measure \mathbb{P} ,

$$\begin{aligned}dX_t^{(1)} &= \mu_1 X_t^{(1)} dt + f_1(Y_t, Z_t) X_t^{(1)} dW_t^{(1)} \\dX_t^{(2)} &= \mu_2 X_t^{(2)} dt + f_2(Y_t, Z_t) X_t^{(2)} dW_t^{(2)} \\dY_t &= \frac{1}{\epsilon} (m_Y - Y_t) dt + \frac{\nu_Y \sqrt{2}}{\sqrt{\epsilon}} dW_t^{(Y)} \\dZ_t &= \delta (m_Z - Z_t) dt + \nu_Z \sqrt{2\delta} dW_t^{(Z)}\end{aligned}$$

where $dW_t^{(1)} dW_t^{(2)} = 0$, f_1 and f_2 are bounded above and below away from 0. $\epsilon > 0$ and $\delta > 0$ are small parameters, corresponding to fast and slow mean-reversion, respectively.

Default Barrier:

$$B_i(t) = K_i e^{\eta_i t}, \quad i = 1, 2.$$

Dynamics under EMM

Under the equivalent martingale measure (EMM) \mathbb{P}^* chosen by the market,

$$dX_t^{(1)} = rX_t^{(1)}dt + f_1(Y_t, Z_t)X_t^{(1)}d\tilde{W}_t^{(1)}$$

$$dX_t^{(2)} = rX_t^{(2)}dt + f_2(Y_t, Z_t)X_t^{(2)}d\tilde{W}_t^{(2)}$$

$$dY_t = \left[\frac{1}{\epsilon}(m_Y - Y_t) + \frac{\nu_Y\sqrt{2}}{\sqrt{\epsilon}}\Lambda_1(Y_t, Z_t) \right] dt + \frac{\nu_Y\sqrt{2}}{\sqrt{\epsilon}}d\tilde{W}_t^{(Y)}$$

$$dZ_t = \left[\delta(m_Z - Z_t) + \nu_Z\sqrt{2\delta}\Lambda_2(Y_t, Z_t) \right] dt + \nu_Z\sqrt{2\delta}d\tilde{W}_t^{(Z)}$$

where $d\tilde{W}_t^{(1)}d\tilde{W}_t^{(2)} = 0$, r is the constant interest rate, and Λ_1 and Λ_2 are combined market prices of risk. They are assumed to be bounded and only dependent on volatility factors (Y_t, Z_t) .

Goal: Joint Survival Probability

$$\begin{aligned} u^{\epsilon, \delta} &\equiv u^{\epsilon, \delta}(t, x_1, x_2, y, z) \\ &\equiv \mathbb{P}^* \left\{ \tau_t^{(1)} > T, \tau_t^{(2)} > T \mid \mathbf{X}_t = (x_1, x_2), Y_t = y, Z_t = z \right\} \end{aligned}$$

where $\tau_t^{(i)} = \inf \left\{ s \geq t \mid X_s^{(i)} \leq B_i(s) \right\}$.

Obviously $u^{\epsilon, \delta}(t, x_1, x_2, y, z) = 0$ if $x_1 \leq B_1(t)$ or $x_2 \leq B_2(t)$.

Therefore we focus on the case where $x_1 > B_1(t)$ and $x_2 > B_2(t)$.

PDE Representation

In terms of partial differential equations (PDE),

$$\mathcal{L}^{\epsilon,\delta} u^{\epsilon,\delta} \equiv \left[\frac{\partial}{\partial t} + \mathcal{L}_{(\mathbf{X},Y,Z)} \right] u^{\epsilon,\delta} = 0, \quad x_1 > B_1(t), x_2 > B_2(t)$$

$$u^{\epsilon,\delta}(t, B_1(t), x_2, y, z) = 0, \quad x_2 > B_2(t)$$

$$u^{\epsilon,\delta}(t, x_1, B_2(t), y, z) = 0, \quad x_1 > B_1(t)$$

$$u^{\epsilon,\delta}(T, x_1, x_2, y, z) = 1, \quad x_1 > B_1(t), x_2 > B_2(t)$$

where $\mathcal{L}_{(\mathbf{X},Y,Z)}$ is the infinitesimal generator of (\mathbf{X}_t, Y_t, Z_t) under measure \mathbb{P}^* .

Operator Decomposition

$$\mathcal{L}^{\epsilon,\delta} = \frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\delta}\mathcal{M}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\epsilon}}\mathcal{M}_3.$$

$$\mathcal{L}_0 = (m_Y - y)\frac{\partial}{\partial y} + \nu_Y^2\frac{\partial^2}{\partial y^2}$$

$$\mathcal{L}_1 = \nu_Y\sqrt{2}\left[\rho_{1Y}f_1x_1\frac{\partial^2}{\partial x_1\partial y} + \rho_{2Y}f_2x_2\frac{\partial^2}{\partial x_2\partial y} - \Lambda_1\frac{\partial}{\partial y}\right]$$

$$\mathcal{L}_2 = \frac{\partial}{\partial t} + \frac{1}{2}f_1^2x_1^2\frac{\partial^2}{\partial x_1^2} + \frac{1}{2}f_2^2x_2^2\frac{\partial^2}{\partial x_2^2} + rx_1\frac{\partial}{\partial x_1} + rx_2\frac{\partial}{\partial x_2}$$

$$\mathcal{M}_1 = \nu_Z\sqrt{2}\left[\rho_{1Z}f_1x_1\frac{\partial^2}{\partial x_1\partial z} + \rho_{2Z}f_2x_2\frac{\partial^2}{\partial x_2\partial z} - \Lambda_2\frac{\partial}{\partial z}\right]$$

Expansion of $u^{\epsilon,\delta}$

$$u^{\epsilon,\delta} = u_0 + \sqrt{\epsilon}u_{1,0} + \sqrt{\delta}u_{0,1} + \dots$$

$\tilde{u} \equiv u_0 + \sqrt{\epsilon}u_{1,0} + \sqrt{\delta}u_{0,1}$ will be our approximation for the true value $u^{\epsilon,\delta}$ — First order approximation.

Term u_0

$$\begin{aligned}\langle \mathcal{L}_2 \rangle u_0 &= 0, & x_1 > B_1(t), x_2 > B_2(t) \\ u_0(t, B_1(t), x_2; z) &= 0, & x_2 > B_2(t) \\ u_0(t, x_1, B_2(t); z) &= 0, & x_1 > B_1(t) \\ u_0(T, x_1, x_2; z) &= 1, & x_1 > B_1(t), x_2 > B_2(t)\end{aligned}$$

where $\langle \cdot \rangle$ denotes the average (in y) with respect to $\mathcal{N}(m_Y, \nu_Y^2)$.

Note that u_0 depends on z in a parametric way.

Define

$$\bar{\sigma}_i(z) = \sqrt{\langle f_i^2(\cdot, z) \rangle}, \quad i = 1, 2.$$

They are **effective volatilities**.

Probabilistic Representation for u_0

$$u_0 = \bar{\mathbb{E}} \left\{ \mathbf{1} \left\{ \inf_{t \leq s \leq T} \bar{X}_s^{(1)} / B_1(s) > 1, \inf_{t \leq s \leq T} \bar{X}_s^{(2)} / B_2(s) > 1 \right\} \middle| \bar{\mathbf{X}}_t = (x_1, x_2) \right\}$$

where under measure $\bar{\mathbb{P}}$,

$$\begin{aligned} d\bar{X}_t^{(1)} &= r\bar{X}_t^{(1)} dt + \bar{\sigma}_1(z)\bar{X}_t^{(1)} d\bar{W}_t^{(1)} \\ d\bar{X}_t^{(2)} &= r\bar{X}_t^{(2)} dt + \bar{\sigma}_2(z)\bar{X}_t^{(2)} d\bar{W}_t^{(2)} \end{aligned}$$

with $d\bar{W}_t^{(1)} d\bar{W}_t^{(2)} = 0$. In other words, $\bar{X}_t^{(1)}$ and $\bar{X}_t^{(2)}$ are independent Geometric Brownian Motions.

Final Solution of u_0

$$u_0 = \left[N\left(d_2^{+(1)}\right) - \left(\frac{x_1}{B_1(t)}\right)^{p_1} N\left(d_2^{-(1)}\right) \right] \\ \times \left[N\left(d_2^{+(2)}\right) - \left(\frac{x_2}{B_2(t)}\right)^{p_2} N\left(d_2^{-(2)}\right) \right]$$

where $N(\cdot)$ is the standard normal cumulative distribution function, and

$$d_2^{\pm(i)} = \frac{\pm \ln(x_i/B_i(t)) + (r - \eta_i - \bar{\sigma}_i^2/2)(T - t)}{\bar{\sigma}_i \sqrt{T - t}} \\ p_i = 1 - \frac{2(r - \eta_i)}{\bar{\sigma}_i^2}.$$

Term $u_{1,0}$

$$\begin{aligned} \langle \mathcal{L}_2 \rangle u_{1,0} &= \mathcal{A}u_0, & x_1 > B_1(t), x_2 > B_2(t) \\ u_{1,0}(t, B_1(t), x_2; z) &= 0, & x_2 > B_2(t) \\ u_{1,0}(t, x_1, B_2(t); z) &= 0, & x_1 > B_1(t) \\ u_{1,0}(T, x_1, x_2; z) &= 0, & x_1 > B_1(t), x_2 > B_2(t) \end{aligned}$$

where $\mathcal{A} = \langle \mathcal{L}_1 \mathcal{L}_0^{-1} (\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) \rangle$. It can be shown that

$$\begin{aligned} \sqrt{\epsilon} \mathcal{A} &= V_{11}^{(2)} x_1^2 \frac{\partial^2}{\partial x_1^2} + V_{22}^{(2)} x_2^2 \frac{\partial^2}{\partial x_2^2} \\ &+ V_{111}^{(3)} x_1 \frac{\partial}{\partial x_1} \left(x_1^2 \frac{\partial^2}{\partial x_1^2} \right) + V_{222}^{(3)} x_2 \frac{\partial}{\partial x_2} \left(x_2^2 \frac{\partial^2}{\partial x_2^2} \right) \\ &+ V_{112}^{(3)} x_2 \frac{\partial}{\partial x_2} \left(x_1^2 \frac{\partial^2}{\partial x_1^2} \right) + V_{122}^{(3)} x_1 \frac{\partial}{\partial x_1} \left(x_2^2 \frac{\partial^2}{\partial x_2^2} \right) \end{aligned}$$

where all V 's are small of order $\sqrt{\epsilon}$ and only dependent on z .

Transformation of $u_{1,0}$

$$v_{1,0} = \sqrt{\epsilon}u_{1,0} + (T - t)\sqrt{\epsilon}\mathcal{A}u_0.$$

Then

$$\begin{aligned}\langle \mathcal{L}_2 \rangle v_{1,0} &= 0, & x_1 > B_1(t), x_2 > B_2(t) \\ v_{1,0}(t, B_1(t), x_2; z) &= g_1(t, x_2), & x_2 > B_2(t) \\ v_{1,0}(t, x_1, B_2(t); z) &= g_2(t, x_1), & x_1 > B_1(t) \\ v_{1,0}(T, x_1, x_2; z) &= 0, & x_1 > B_1(t), x_2 > B_2(t)\end{aligned}$$

where

$$\begin{aligned}g_1(t, x_2) &= (T - t) \lim_{x_1 \downarrow B_1(t)} \sqrt{\epsilon}\mathcal{A}u_0 \\ g_2(t, x_1) &= (T - t) \lim_{x_2 \downarrow B_2(t)} \sqrt{\epsilon}\mathcal{A}u_0\end{aligned}$$

Change of Variables

Let $\{\xi_s^{(i)}\}_{t \leq s \leq T}$ be two independent Brownian Motions with drifts $\bar{\mu}_i = (r - \eta_i - \bar{\sigma}_i^2/2)/\bar{\sigma}_i$ under measure $\bar{\mathbb{P}}$, and

$$\xi_t^{(i)} = \xi_i = \frac{1}{\bar{\sigma}_i} \ln(x_i/B_i(t)).$$

Denote

$$\begin{aligned} v_{1,0}(t, x_1, x_2; z) &= \bar{v}_{1,0}(t, \xi_1, \xi_2; z) \\ g_1(t, x_2) &= \bar{g}_1(t, \xi_2) \\ g_2(t, x_1) &= \bar{g}_2(t, \xi_1) \end{aligned}$$

Probabilistic Representation of $\bar{v}_{1,0}$

$$\begin{aligned} \bar{v}_{1,0} = & \mathbb{E} \left\{ \bar{g}_1(\bar{\tau}, \xi_{\bar{\tau}}^{(2)}) \mathbf{1}_{(\bar{\tau}=\bar{\tau}_1)} \mathbf{1}_{(\bar{\tau} \leq T)} \right. \\ & \left. + \bar{g}_2(\bar{\tau}, \xi_{\bar{\tau}}^{(1)}) \mathbf{1}_{(\bar{\tau}=\bar{\tau}_2)} \mathbf{1}_{(\bar{\tau} \leq T)} \middle| \xi_t^{(1)} = \xi_1, \xi_t^{(2)} = \xi_2 \right\} \end{aligned}$$

for $\xi_1 > 0$ and $\xi_2 > 0$, where

$$\begin{aligned} \bar{\tau}_i &= \inf \left\{ s \geq t \mid \xi_s^{(i)} \leq 0 \right\}, \quad i = 1, 2. \\ \bar{\tau} &= \min\{\bar{\tau}_1, \bar{\tau}_2\}. \end{aligned}$$

Two Probability Densities

$$\begin{aligned}
 & \bar{\mathbb{P}} \left\{ \bar{\tau} \in ds, \bar{\tau} = \bar{\tau}_1, \xi_{\bar{\tau}}^{(2)} \in d\xi \mid \xi_t^{(1)} = \xi_1, \xi_t^{(2)} = \xi_2 \right\} \\
 &= \frac{\xi_1}{\pi(s-t)^2} \exp \left\{ \bar{\mu}_2 \xi - \frac{\xi^2}{2(s-t)} \right\} \sinh \left(\frac{\xi \xi_2}{2(s-t)} \right) \\
 & \exp \left\{ -\frac{\xi_1^2 + \xi_2^2}{2(s-t)} - \bar{\mu}_1 \xi_1 - \bar{\mu}_2 \xi_2 - (\bar{\mu}_1^2 + \bar{\mu}_2^2)(s-t)/2 \right\} d\xi ds
 \end{aligned}$$

$\bar{\mathbb{P}} \left\{ \bar{\tau} \in ds, \bar{\tau} = \bar{\tau}_2, \xi_{\bar{\tau}}^{(1)} \in d\xi \mid \xi_t^{(1)} = \xi_1, \xi_t^{(2)} = \xi_2 \right\}$ can be obtained by symmetry.

Term $u_{0,1}$

$$\begin{aligned}\langle \mathcal{L}_2 \rangle u_{0,1} &= -\langle \mathcal{M}_1 \rangle u_0, & x_1 > B_1(t), x_2 > B_2(t) \\ u_{0,1}(t, B_1(t), x_2; z) &= 0, & x_2 > B_2(t) \\ u_{0,1}(t, x_1, B_2(t); z) &= 0, & x_1 > B_1(t) \\ u_{0,1}(T, x_1, x_2; z) &= 0, & x_1 > B_1(t), x_2 > B_2(t)\end{aligned}$$

It can be shown that

$$\begin{aligned}\sqrt{\delta} \langle \mathcal{M}_1 \rangle u_0 &= V_1^{(0)} \frac{\partial u_0}{\partial \sigma_1} + V_2^{(0)} \frac{\partial u_0}{\partial \sigma_2} \\ &+ V_{11}^{(1)} x_1 \frac{\partial}{\partial x_1} \left(\frac{\partial u_0}{\partial \sigma_1} \right) + V_{12}^{(1)} x_1 \frac{\partial}{\partial x_1} \left(\frac{\partial u_0}{\partial \sigma_2} \right) \\ &+ V_{21}^{(1)} x_2 \frac{\partial}{\partial x_2} \left(\frac{\partial u_0}{\partial \sigma_1} \right) + V_{22}^{(1)} x_2 \frac{\partial}{\partial x_2} \left(\frac{\partial u_0}{\partial \sigma_2} \right)\end{aligned}$$

where all V 's are small of order $\sqrt{\delta}$ and only dependent on z .

Transformation of $u_{0,1}$

Define operators M_1 and M_2 such that

$$\sqrt{\delta}\langle\mathcal{M}_1\rangle u_0 = M_1 \frac{\partial u_0}{\partial \sigma_1} + M_2 \frac{\partial u_0}{\partial \sigma_2}.$$

Now define

$$\begin{aligned} v_{0,1} = & \sqrt{\delta}u_{0,1} - (T - t) \left(M_1 \frac{\partial u_0}{\partial \sigma_1} + M_2 \frac{\partial u_0}{\partial \sigma_2} \right) \\ & + \frac{1}{2}(T - t)^2 \left(M_1 \bar{\sigma}_1 x_1^2 \frac{\partial^2 u_0}{\partial x_1^2} + M_2 \bar{\sigma}_2 x_2^2 \frac{\partial^2 u_0}{\partial x_2^2} \right) \end{aligned}$$

PDE for $v_{0,1}$

$$\begin{aligned}
 \langle \mathcal{L}_2 \rangle v_{0,1} &= 0, & x_1 > B_1(t), x_2 > B_2(t) \\
 v_{0,1}(t, B_1(t), x_2; z) &= g_3(t, x_2), & x_2 > B_2(t) \\
 v_{0,1}(t, x_1, B_2(t); z) &= g_4(t, x_1), & x_1 > B_1(t) \\
 v_{0,1}(T, x_1, x_2; z) &= 0, & x_1 > B_1(t), x_2 > B_2(t)
 \end{aligned}$$

where

$$\begin{aligned}
 g_1(t, x_2) &= -(T-t) \lim_{x_1 \downarrow B_1(t)} \left(M_1 \frac{\partial u_0}{\partial \sigma_1} + M_2 \frac{\partial u_0}{\partial \sigma_2} \right) \\
 &\quad + \frac{1}{2} (T-t)^2 \lim_{x_1 \downarrow B_1(t)} \left(M_1 \bar{\sigma}_1 x_1^2 \frac{\partial^2 u_0}{\partial x_1^2} + M_2 \bar{\sigma}_2 x_2^2 \frac{\partial^2 u_0}{\partial x_2^2} \right) \\
 g_2(t, x_1) &= -(T-t) \lim_{x_2 \downarrow B_2(t)} \left(M_1 \frac{\partial u_0}{\partial \sigma_1} + M_2 \frac{\partial u_0}{\partial \sigma_2} \right) \\
 &\quad + \frac{1}{2} (T-t)^2 \lim_{x_2 \downarrow B_2(t)} \left(M_1 \bar{\sigma}_1 x_1^2 \frac{\partial^2 u_0}{\partial x_1^2} + M_2 \bar{\sigma}_2 x_2^2 \frac{\partial^2 u_0}{\partial x_2^2} \right)
 \end{aligned}$$

Solution for $v_{0,1}$

By doing the same type of change of variables as in the case of $v_{1,0}$, $v_{1,0}$ can be represented as an expectation, which is essentially a double integral.

Model Calibration (ongoing work)

- Calibrate to individual default probability (or yield spread, CDS data, etc.) to obtain estimates for those parameters entirely determined by individual firms.
- Calibrate to default correlation (or implied correlation from Gaussian copula) to obtain estimates for those parameters solely connected to the dependency structure.

The End
Thank You