A Jump to Default Extended CEV Model: An Application of Bessel Processes[*]

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SCOPE

- Introduction.
- Jump to Default Extended Diffusion Process.
- Unified Valuation of Corporate Liabilities, Credit Derivatives and Equity Derivatives.
- Jump to Default Extended CEV Stock Price Process.
- Solution to CEV Model.
- Asymptotic Analysis.
- Results.

INTRODUCTION

- Unification of credit derivatives,equity derivatives and valuation of corporate liabilities.
- The model to capture several fundamental empirical observations is both flexible and analytically tractable.
- The analytical tractability of the model stems from the close connection of stock prices to Bessel processes.
- This connection leads to explicit closed form formulas for risk neutral probabilities and arbitrage-free values of corporate bonds, credit derivatives and equity options.

Fundamental Empirical Observations

- CDS spreads and Corporate bond yields are both positively related to implied volatilities of equity options.
- Realized volatility of a stock is negatively related to its price level (leverage effect).
- Similarly, equity implied volatilities tend to be decreasing convex functions of the option's strike price (skew)……..(CEV needed for this).
- The proposed model will capture all the empirical regularities.

Empirical evidence in support of…

- Campbell and Tasher (2003)[12] (*positive relationship between default probabilities and equity volatility*).
- Cremers et al. (2004)[20] (Link between *CDS spreads and equity volatilities*).
- Black(1976)[5] (*Realized stock volatility is negatively related to stock price..leverage effect*).

Jump to Default Extended Diffusion

- Model price of a defaultable stock as time-inhomogeneous diffusion process.
- Notations:
- $\{S_t^{\Delta}, t \geq 0\}$---Defaultable stock process.
- $r(t) \geq 0$ ---Time-dependent risk-free rate.
- $q(t) \geq 0$, ---Time-dependent dividend yield.
- $\sigma(S,t) > 0$---Time- and state –dependent instantaneous stock volatility.
- $\lambda(S,t) \geq 0$ ---Time- and state -dependent default intensity.

Jump to Default Extended Diff……..

- We have a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$ carrying a standard BM $\{B_t, t \geq 0\}$ and an exponential random variable with unit parameter $e \sim \mathrm{Exp}(1)$ independent of B.
- Assume: frictionless markets, no arbitrage and EMM $\mathbb{Q}$ as given, then we model pre-default stock dynamics under EMM as time – inhomogeneous diffusion process by solving the following SDE :---

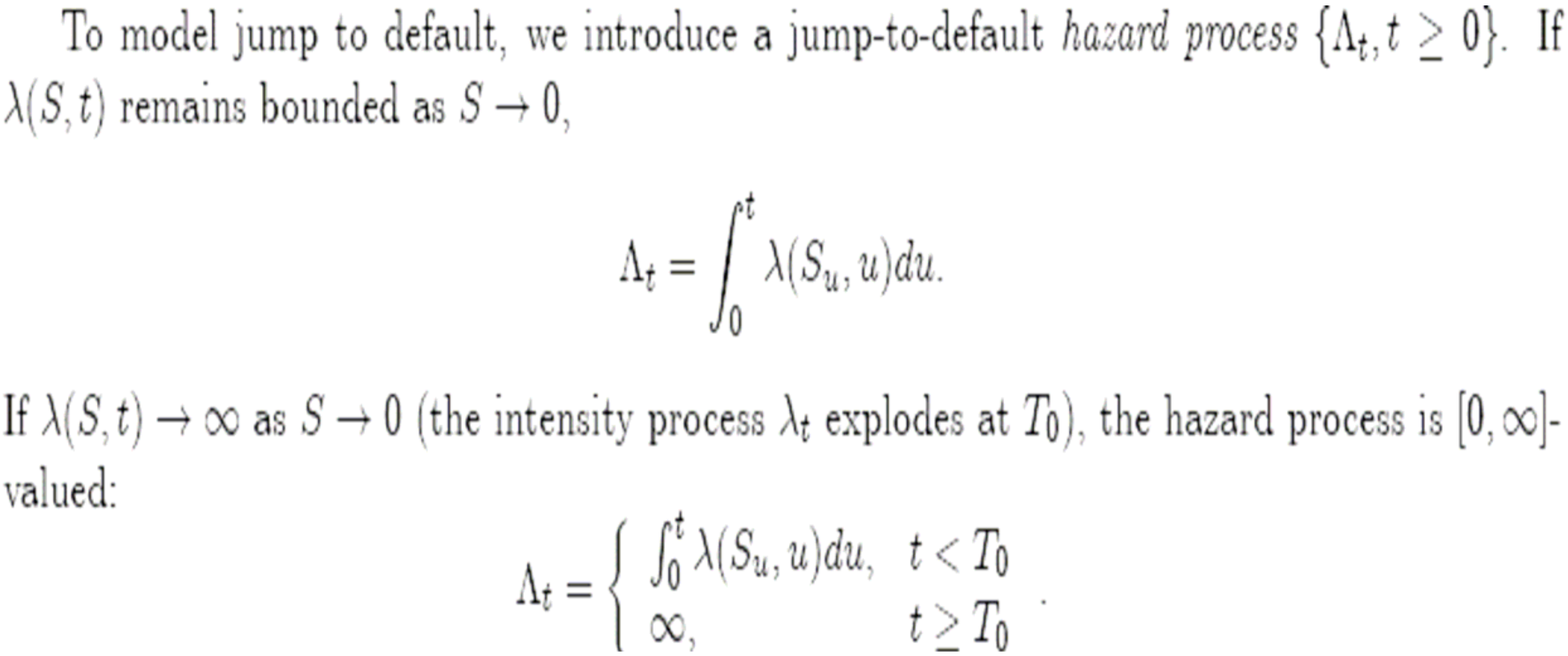
Jump to Default Extended Diff…..

- $$dS_t = [r(t) - q(t) + \lambda(S_t, t)]S_t\,dt + \sigma(S_t, t)S_t\,dB_t, \quad S_0 = S > 0, \qquad (2.1)$$
- $q(t)$ can be adapted from dividend forecast.
- $\sigma(S,t)$ and $\lambda(S,t)$ can be inferred from a complete term and strike structure of implied equity volatilities.
- Assumed $\sigma(S,t)$ and $\lambda(S,t)$ is continuously differentiable c $[0,$

$\lambda(S,t)$ and $\sigma(S,t)$ are continuously differentiable in state and time $(0,\infty) \times [0,\infty)$

In the terminology of Markov processes, if zero is an exit or a regular killing boundary, at the first hitting time of zero, $T_0 = \inf\{t \geq 0 : S_t = 0\}$, S is sent to the *cemetery state* Δ, where it remains forever (Δ is an absorbing state)

Jump to Default Extended Diff……



To model jump to default, we introduce a jump-to-default *hazard process* $\{\Lambda_t, t \geq 0\}$. If $\lambda(S,t)$ remains bounded as $S \to 0$,

$$\Lambda_t = \int_0^t \lambda(S_u, u)du.$$

If $\lambda(S,t) \to \infty$ as $S \to 0$ (the intensity process λ_t explodes at T_0), the hazard process is $[0,\infty]$-valued:

$$\Lambda_t = \begin{cases} \int_0^t \lambda(S_u, u)du, & t < T_0 \\ \infty, & t \geq T_0 \end{cases}.$$

Jump to Default Extended Diff…..

We model the random time of jump to default $\tilde{\zeta}$ as the first time when the hazard process Λ is greater or equal to the random level $e \sim \mathrm{Exp}(1)$:

$$\tilde{\zeta} = \inf\{t \geq 0 : \Lambda_t \geq e\}.$$

At time $\tilde{\zeta}$, the stock jumps to the cemetery (bankruptcy) state, Δ, where it remains forever.[1] We assume equity holders do not receive any recovery in the event of bankruptcy and their equity position becomes worthless. We denote the defaultable stock process $S^\Delta = \{S_t^\Delta, t \geq 0\}$. We note that, in general, in this model default can happen either at time T_0 via diffusion to zero or at time $\tilde{\zeta}$ via a jump to default, whichever comes first. The time of default ζ (*lifetime* of the process S^Δ in the terminology of Markov processes) is then

$$\zeta = T_0 \wedge \tilde{\zeta}.$$

3

Unified Valuation of Corporate Liabilities, Credit Derivatives, and Equity Derivatives

Risk neutral probability of surviving:

Conditioning on the information available at time $t \geq 0$, the risk-neutral probability of surviving beyond some fixed time $T > t$ is given by:

$$\mathbb{Q}(\zeta > T|\mathcal{G}_t) = \mathbf{1}_{\{\zeta>t\}}\mathbb{E}[e^{-\int_t^T \lambda(S_u,u)du}\mathbf{1}_{\{T_0>T\}}|\mathcal{F}_t] = \mathbf{1}_{\{\zeta>t\}}Q(S_t,t;T),$$

With the following notation:

$$Q(S,t;T) = \mathbb{E}[e^{-\int_t^T \lambda(S_u,u)du}\mathbf{1}_{\{T_0>T\}}|S_t = S]. \qquad (3.1)$$

$$Given\!: \zeta = T_0 \wedge \tilde{\zeta} \; and \, \mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$$

$$\mathsf{Q}(\zeta > T \mid \mathcal{G}_t) = \mathsf{Q}\left(\left(T_0 \wedge \tilde{\zeta}\right) > T \mid \mathcal{G}_t\right) = \mathsf{Q}(T_0 > T, \tilde{\zeta} > T \mid \mathcal{G}_t) = E\left[\mathbf{1}_{\{T_0>T\}}\mathbf{1}_{\{\tilde{\zeta}>T\}}\middle|\mathcal{G}_t\right]$$

$$= \mathbf{1}_{\{\zeta>t\}}E\left[\mathbf{1}_{\{T_0>T\}}\mathbf{1}_{\{\tilde{\zeta}>T\}}\middle|\mathcal{F}_t \vee \{\zeta > t\}\right] = \mathbf{1}_{\{\zeta>t\}}E\left[E\left[\mathbf{1}_{\{T_0>T\}}\mathbf{1}_{\{\tilde{\zeta}>T\}}\middle|\mathcal{F}_T \vee \{\zeta > t\}\right]\middle|\mathcal{F}_t \vee \{\zeta > t\}\right]$$

$$= \mathbf{1}_{\{\zeta>t\}}E\left[\mathbf{1}_{\{T_0>T\}}E\left[\mathbf{1}_{\{\tilde{\zeta}>T\}}\middle|\mathcal{F}_T \vee \{\zeta > t\}\right]\middle|\mathcal{F}_t \vee \{\zeta > t\}\right]$$

$$= \mathbf{1}_{\{\zeta>t\}}E\left[\mathbf{1}_{\{T_0>T\}}\mathsf{Q}\left(\tilde{\zeta} > T\middle|\mathcal{F}_T \vee \{\zeta > t\}\right)\middle|\mathcal{F}_t \vee \{\zeta > t\}\right] = \mathbf{1}_{\{\zeta>t\}}E\left[e^{-\int_t^T \lambda(S_u,u)du}\mathbf{1}_{\{T_0>T\}}\middle|\mathcal{F}_t\right]$$

Three simple building block claims:

(i) A European-style contingent claim with maturity (expiration) at time $T > 0$ and payoff $\psi(S_T)$ at T, given no default by T, and no recovery if default happens by T;

$$
\begin{aligned}
& e^{-\int_t^T r(u)du}\mathbb{E}\left[\Psi(S_T)\mathbf{1}_{\{\zeta>T\}}\middle|\mathcal{G}_t\right] \\
&= \mathbf{1}_{\{\zeta>t\}}e^{-\int_t^T r(u)du}\mathbb{E}\left[e^{-\int_t^T \lambda(S_u,u)du}\Psi(S_T)\mathbf{1}_{\{T_0>T\}}\middle| S_t\right]; \qquad (3.2)
\end{aligned}
$$

Three simple building block claims: (cont'd)

(ii) A recovery payment of one dollar paid at the maturity date T if default occurs by T;

$$e^{-\int_t^T r(u)du}\mathbb{E}\left[\mathbf{1}_{\{\zeta\le T\}}\,\middle|\,\mathcal{G}_t\right] = e^{-\int_t^T r(u)du}[1-\mathbf{1}_{\{\zeta>t\}}Q(S_t,t;T)]; \tag{3.3}$$

Three simple building block claims: (cont’d)

(iii) A recovery payment of one dollar paid at the default time ζ

$$
\begin{aligned}
&\mathbb{E}\left[e^{-\int_t^{\zeta} r(u)du} \mathbf{1}_{\{\zeta \le T\}} \middle| \mathcal{G}_t \right] \\
&= \mathbf{1}_{\{\zeta > t\}} \mathbb{E}\left[\int_t^T e^{-\int_t^u [r(v)+\lambda(S_v,v)]dv} \lambda(S_u,u) \mathbf{1}_{\{T_0 > u\}} du \middle| \mathcal{F}_t \right] \\
&= \mathbf{1}_{\{\zeta > t\}} \int_t^T e^{-\int_t^u r(v)dv} \mathbb{E}\left[e^{-\int_t^u \lambda(S_v,v)dv} \lambda(S_u,u) \mathbf{1}_{\{T_0 > u\}} \middle| S_t \right] du. \qquad (3.4)
\end{aligned}
$$

Three simple building block claims: (cont'd)

These valuations all reduce to computing risk-neutral expectations of the form:

$$\mathbb{E}\left[e^{-\int_t^T \lambda(S_u,u)du}\,\Psi(S_T)\mathbf{1}_{\{T_0>T\}} \,\middle|\, S_t = S\right] \tag{3.5}$$

Three simple building block claims: (cont'd)

These three building blocks can be used to value corporate liabilities, credit derivatives, and equity derivatives.

For fixed $T > 0$, a *defaultable zero-coupon bond* with unit face value and no recovery can be represented as the European claim with $\Psi(S_T) = 1$,
For time $t < T$, given no default by time t, a *defaultable zero-coupon bond* with unit face value and no recovery can be represented as the discounted risk-neutral survival probability

$$B(S,t;T) = e^{-\int_t^T r(u)du} Q(S,t;T). \qquad (3.6)$$

Three simple building block claims: (cont'd)

A European call option with strike $K > 0$ with the payoff $(S_T - K)^+$ at expiration T has no recovery if the firm defaults.

Assume no default by time $t \in [0, T)$, the pricing formula for call option at time t is:

$$C(S, t; K, T) = e^{-\int_t^T r(u)du} \mathbb{E}\left[e^{-\int_t^T \lambda(S_u, u)du} (S_T - K)^+ \mathbf{1}_{\{T_0 > T\}} \middle| S_t = S \right], \qquad (3.7)$$

Three simple building block claims: (cont'd)

A European put option with strike $K > 0$ with the payoff $(K - S_T)^+$ can be decomposed into two parts:

1. The put payoff $(K - S_T)^+ 1_{\{\zeta > T\}}$, given no default by time T
2. A recovery payment equal to the strike K at expiration in the event of default $\zeta \le T$

Assume no default by time $t \in [0,T)$, the pricing formula for put option at time t is:

$$P(S,t;K,T) = P_0(S,t;K,T) + P_D(S,t;K,T) \tag{3.8}$$

$$= e^{-\int_t^T r(u)du} \mathbb{E}\left[e^{-\int_t^T \lambda(S_u,u)du} (K - S_T)^+ \mathbf{1}_{\{T_0 > T\}} \middle| S_t = S \right] \tag{3.9}$$

$$+Ke^{-\int_t^T r(u)du}[1 - Q(S,t;T)], \tag{3.10}$$

Where $P_0(S,t;K,T)$ is the put of conditional on no default

$P_D(S,t;K,T)$ is the cash payment K in the event of default

4

A Jump to Default Extended CEV Stock Price Process

CEV:

To be consistent with the leverage effect and the implied volatility skew, we specify the instantaneous volatility as that of a constant elasticity of variance (CEV) process:

$$\sigma(S,t) = a(t)S^{\beta}, \tag{4.1}$$

Where $\beta<0$ is the volatility elasticity parameter and $a(t) > 0$ is the time-dependent volatility scale parameter

JDCEV:

To be consistent with the empirical evidence linking bond yields and CDS spreads to equity volatility, we specify the default intensity as an affine function of the instantaneous variance of the underlying stock:

$$\lambda(S,t) = b(t) + c\,\sigma^2(S,t) = b(t) + c\,a^2(t)S^{2\beta}, \qquad (4.2)$$

Where $b(t) \geq 0$ is a deterministic non-negative function of time and $c > 0$ is positive constant parameter governing the sensitivity of λ to σ^2

The function of $b(t)$ and $a(t)$ can be determined by reference to given term structures of CDS spreads and at-the-money implied volatilities

JDCEV: (cont'd)

For $c \geq 1/2$, the SDE(2.1) with σ and λ specified by (4.1-2) has a unique non-exploding solution. This solution is a diffusion process on $(0,\infty)$ where zero and infinity are both unattainable boundaries.

For $c \geq 1/2$, $T_0 = \infty$, since zero is an unattainable boundary.

JDCEV: (cont'd)

For $c \in (0,1/2)$, infinity is an unattainable (natural) boundary for $\beta < 0$, while zero is an exit boundary for $\beta \in [c - 1/2, 0)$ For $c \in (0,1/2)$ and $\beta < c - 1/2$, zero is a regular boundary, and we specify it as killing boundary by adjoining a killing boundary condition. Thus for $c \in (0,1/2)$ at T_0 the process S is sent to the cemetery state Δ.

Even though $c \in (0,1/2)$, the pre-default process S can hit zero, since the intensity $\lambda(S,t)$ goes to infinity as S goes to zero, zero is an unattainable boundary for the process S^{Δ} killed at rate $\lambda(S,t)$ ($\tilde{\zeta} < T_0$ a.s. and $\zeta = \tilde{\zeta}$ a.s.)

JDCEV: (cont'd)

To summarize the specification, under the EMM, we model the price of the defaultable stock as a time-inhomogeneous diffusion process $\{S_t^\Delta, t \geq 0\}$ with state space $E^\Delta = (0,\infty) \cup \{\Delta\}$, initial value $S_0 = x > 0$, diffusion coefficient $a(t)x^{\beta+1}$, drift $[r(t) - q(t) + b(t) + ca^2(t)x^{2\beta}]x$, and killing rate $b(t) + ca^2(t)x^{2\beta}$.

We refer to the stock price as the *jump to default extended CEV* process, or *JDCEV* for short.

Remark 4.1. Advantage of JDCEV

Merton(1976) , Jarrow and Turnbull(1995):
Constant arrival rate λ and constant volatility σ

1. This model is tractable and produces downward slopping implied volatility skews.
2. It's also straightforward to extend this model to deterministically time varying default arrival rates and instantaneous volatilities, so as to accommodate observed term structures of CDS and at-the-money implied volatilities.
3. CDS spreads will not behave randomly, and in particular will be independent of stock price movements.
4. The determinacy of volatility is inconsistent with both stochastic volatility and the leverage effect.

Remark 4.1. Advantage of JDCEV(cont'd)

Madan and Unal(1998) , Linetsky(2005):
Analytically tractable specifications for the default intensity function $\lambda(S,t)$

1. The instantaneous stock volatility remains constant so that the stochastic volatility and the leverage effect remain unaddressed.
2. The risk of default is the sole determinant of the implied volatility skew.
3. In practice, the skew tends to flatten out as maturity increases and these time homogeneous models are unable to capture the magnitude of the skew at both long and short maturities.

Remark 4.1. Advantage of JDCEV(cont’d)

Campi et al.(2005):
CEV specification for variance, but time homogeneity

1. Assume constant arrival rate of default which leads to the implication that short term credit spreads have low (but not zero) sensitivity to stock price levels and implied volatilities.

Remark 4.1. Advantage of JDCEV(cont'd)

JDCEV:

1. The hazard rate and the instantaneous variance both depend on the stock price
2. Accommodate large negative correlations between default indicators and stock prices, and between realized volatilities and stock prices.
3. Include the large positive correlation between default indicators and volatilities that have been observed in the market.
4. The parameters β and c both play a role in determining the slope of the implied volatility skew, which gives more flexibility in accommodating slopes which vary with term.

Remark 4.2. The JDCEV SDE

For special case, $c=1$ and constant parameters, this process was discussed in Heath and Platen(2002), Delbean and Shirakawa (2002)

For $c \in \mathrm{R}$ and constant parameters, it was extensively studied in the monograph in Lehnigk(1993) using PDE methods. In this monograph, it is called the *generalized Feller process* as it nests Feller's (1951) square-root diffusion as a special case with $\beta=-1/2$

Section 5: Solutions of Jump to Default Extended CEV Model via the Theory of Bessel Processes

Bessel Process

Let $\{R_t^{(\nu)}, t \geq 0\}$ be a Bessel process of order ν and started at $x > 0$: $BES^{(\nu)}(x)$.

- $\nu \geq 0$: zero is an unattainable boundary.
- $\nu < 0$: zero is attainable. We specify zero as a killing boundary and send the Bessel process with $\nu < 0$ to the cemetery state Δ at the first hitting time of zero, $T_0^R = \inf\{t \geq 0 : R_t^{(\nu)} = 0\}$.

Connection between stock price and Bessel process

Proposition 5.1 *Let* $\{S_t, t \geq 0\}$ *be the process evolving according to (2.1) with* σ *and* λ *specified in (4.1-2). For* $c \geq 1/2$*, the process (2.1) can be represented as a re-scaled and time-changed power of a Bessel process:*

$$\{S_t = \mathrm{e}^{\int_0^t \alpha(u)\mathrm{d}u}(|\beta|\, R^{(\nu)}_{\tau(t)})^{\frac{1}{|\beta|}}, t \geq 0\},$$

where:

$$\tau(t) = \int_0^t a^2(u)\mathrm{e}^{-2|\beta|\int_0^u \alpha(s)\mathrm{d}s}\mathrm{d}u,\ \nu = \frac{c - 1/2}{|\beta|} \in \mathbb{R},$$

$$R^{(\nu)}_0 = x = \frac{1}{|\beta|}S^{|\beta|} > 0,\ \alpha(t) = r(t) - q(t) + b(t).$$

For $c \in (0, 1/2)(\nu < 0)$*, the same representation holds before the first hitting time of zero,* $T^S_0 = \tau^{-1}(T^R_0)$ $(T^R_0 = \tau(T^S_0))$.

Reduce computing risk-neutral expectation

Proposition 5.2 *For any* $0 \leq t < T$*:*

$$\begin{aligned}
&\mathbb{E}\left[\exp\left\{-c\int_t^T a^2(u) S_u^{2\beta} \mathrm{d}u\right\}\Psi(S_T)\mathbf{1}_{\{T_0^S > T\}}\middle| S_t = S\right] \\
= \;& E_x^{(\nu)}\left[\exp\left\{-\frac{c}{\beta^2}\int_0^\tau \frac{\mathrm{d}u}{R_u^2}\right\}\Psi\left(\mathrm{e}^{\int_t^T \alpha(s)\mathrm{d}s}(|\beta|\, R_\tau)^{\frac{1}{|\beta|}}\right)\mathbf{1}_{\{T_0^R > \tau\}}\right],
\end{aligned}$$

where $E_x^{(\nu)}$ *denotes expectation calculated with respect to the law of* $BES^{(\nu)}(x)$*. Here* ν *and* x *are as in Proposition 5.1, and*

$$\tau = \tau(t,T) = \int_t^T a^2(u)\mathrm{e}^{-2|\beta|\int_t^u \alpha(s)\mathrm{d}s}\mathrm{d}u.$$

Absolute continuity between two Bessel Processes

Proposition 5.3 *Let* $P_x^{(\nu)}$ *be the law of the Bessel process* $R^{(\nu)}$ *started at* $x > 0$ *and let* $\mathcal{R}_t$ *be its canonical filtration. For* $\nu \geq 0$ *and* $\mu \geq 0$ *the following absolute continuity relation holds:*

$$P_x^{(\nu)}|_{\mathcal{R}_t} = \left(\frac{R_t}{x}\right)^{\nu-\mu} \exp\left(-\frac{\nu^2-\mu^2}{2}\int_0^t \frac{\mathrm{d}u}{R_u^2}\right) P_x^{(\mu)}|_{\mathcal{R}_t}.$$

For $\nu < 0$ *and* $\mu \geq 0$ *the following relation holds before the first hitting time of zero* T_0^R*:*

$$P_x^{(\nu)}|_{\mathcal{R}_t \cap \{t < T_0^R\}} = \left(\frac{R_t}{x}\right)^{\nu-\mu} \exp\left(-\frac{\nu^2-\mu^2}{2}\int_0^t \frac{\mathrm{d}u}{R_u^2}\right) P_x^{(\mu)}|_{\mathcal{R}_t}.$$

Further reduce computing risk-neutral expectation

Proposition 5.4

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left\{-c\int_t^T a^2(u)S_u^{2\beta}\mathrm{d}u\right\}\Psi(S_T)\mathbf{1}_{\{T_0^S>T\}}\middle| S_t=S\right]\\
=\;\; & E_x^{(\nu_+)}\left[\left(\frac{R_\tau}{x}\right)^{-\frac{1}{|\beta|}}\Psi\left(\mathrm{e}^{\int_t^T\alpha(s)\mathrm{d}s}(|\beta|\,R_\tau)^{\frac{1}{|\beta|}}\right)\right],
\end{aligned}
$$

where

$$\nu_+=\nu+\frac{1}{|\beta|}=\frac{c+1/2}{|\beta|}>0.$$

Relationship between Bessel process and non-central chi-square

- For $\nu \geq 0$ the density of $R_\tau > 0$ started at $R_0 = x > 0$ is

$$p^{(\nu)}(\tau; x, y) = \frac{y}{\tau}\left(\frac{y}{x}\right)^{\nu} \exp\left(-\frac{x^2 + y^2}{2\tau}\right) I_\nu\left(\frac{xy}{\tau}\right),$$

where $I_\nu(z)$ is the modified Bessel function of order ν.

- The density of non-central chi-square $\chi^2(\delta, \alpha)$ with δ degrees of freedom and non-centrality parameter $\alpha > 0$ is

$$f_{\chi^2}(x; \delta, \alpha) = \frac{1}{2} e^{-\frac{\alpha+x}{2}} \left(\frac{x}{\alpha}\right)^{\frac{\nu}{2}} I_\nu(\sqrt{x\alpha}) \mathbf{1}_{\{x>0\}},$$

where $\nu = \delta/2 - 1$.

- Relation

$$p^{(\nu)}(\tau; x, y) = \left(\frac{2y}{\tau}\right) f_{\chi^2}\left(\frac{y^2}{\tau}; \delta, \frac{x^2}{\tau}\right).$$

Moments of non-central chi-square distribution

Lemma 5.1 *Let* X *be a* $\chi^2(\delta,\alpha)$ *random variable,* $\nu = \delta/2 - 1$, $p > -(\nu+1)$ *and* $k > 0$. *The (truncated)* p*-th moments are*

$$\mathcal{M}(p;\delta,\alpha) = E^{\chi^2(\delta,\alpha)}[X^p] = 2^p e^{-\frac{\alpha}{2}} \frac{\Gamma(p+\nu+1)}{\Gamma(\nu+1)} {}_1F_1(p+\nu+1, \nu+1, \alpha/2),$$

$$\Phi^+(p,k;\delta,\alpha) = E^{\chi^2(\delta,\alpha)}[X^p \mathbf{1}_{\{X>k\}}] = 2^p e^{-\frac{\alpha}{2}} \sum_{n=0}^{\infty} \frac{\Gamma(\nu+p+n+1, k/2)}{n!\Gamma(\nu+n+1)},$$

$$\Phi^-(p,k;\delta,\alpha) = E^{\chi^2(\delta,\alpha)}[X^p \mathbf{1}_{\{X\leq k\}}] = 2^p e^{-\frac{\alpha}{2}} \sum_{n=0}^{\infty} \frac{\gamma(\nu+p+n+1, k/2)}{n!\Gamma(\nu+n+1)},$$

where $\gamma(a,x) = \int_0^x y^{a-1} e^{-y} dy$, $\Gamma(a,x) = \Gamma(a) - \gamma(a,x)$, *and*

$${}_1F_1(a,b,x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{x^n}{n!}, where (a)_0 = 1, (a)_n = a(a+1)\ldots(a+n-1).$$

Final Solutions of JDCEV Model

Proposition 5.5 *Let* $\delta_+ = 2(\nu_+ + 1)$*. Assume that default has not happened by time* $t \geq 0$*. (i) The risk-neutral survival probability is given by*

$$Q(S,t;T) = \mathrm{e}^{-\int_t^T b(u)\mathrm{d}u} \left(\frac{x^2}{\tau}\right)^{\frac{1}{2|\beta|}} \mathcal{M}\left(-\frac{1}{2|\beta|}; \delta_+, \frac{x^2}{\tau}\right).$$

(ii) The claim that pays one dollar at the time of default is

$$\int_t^T \mathrm{e}^{-\int_t^u [r(s)+b(s)]\mathrm{d}s} \left\{ b(u) \left(\frac{x^2}{\tau(t,u)}\right)^{\frac{1}{2|\beta|}} \mathcal{M}\left(-\frac{1}{2|\beta|}; \delta_+, \frac{x^2}{\tau(t,u)}\right)\right.$$

$$\left. + cS^{2\beta} a^2(u) \mathrm{e}^{-2|\beta| \int_t^u \alpha(s)\mathrm{d}s} \left(\frac{x^2}{\tau(t,u)}\right)^{\frac{1}{2|\beta|}+1} \mathcal{M}\left(-\frac{1}{2|\beta|} - 1; \delta_+, \frac{x^2}{\tau(t,u)}\right)\right\} \mathrm{d}u.$$

(iii) The call option price is given by

$$C(S,t;K,T) = \mathrm{e}^{-\int_t^T q(u)\mathrm{d}u} S\Phi^+\left(0, \frac{k^2}{\tau}; \delta_+, \frac{x^2}{\tau}\right)$$

$$-\mathrm{e}^{-\int_t^T [r(u)+b(u)]\mathrm{d}u} K \left(\frac{x^2}{\tau}\right)^{\frac{1}{2|\beta|}} \Phi^+\left(-\frac{1}{2|\beta|}, \frac{k^2}{\tau}; \delta_+, \frac{x^2}{\tau}\right),$$

where

$$k = k(t,T) = \frac{1}{|\beta|} K^{|\beta|} \mathrm{e}^{-|\beta| \int_t^T \alpha(u)\mathrm{d}u}.$$

(iv) The price of the put payoff conditional on no default by time T is given by

$$\begin{aligned} P_0(S,t;K,T) &= e^{-\int_t^T [r(u)+b(u)]du} K \left(\frac{x^2}{\tau}\right)^{\frac{1}{2|\beta|}} \Phi^- \left(-\frac{1}{2|\beta|}, \frac{k^2}{\tau}; \delta_+, \frac{x^2}{\tau}\right) \\ &- e^{-\int_t^T q(u)du} S \Phi^- \left(0, \frac{k^2}{\tau}; \delta_+, \frac{x^2}{\tau}\right). \end{aligned}$$

The recovery part of the put option is given by

$$P_D(S,t;K,T) = K e^{-\int_t^T r(u)du}[1 - Q(S,t;T)].$$

Put-Call parity

$$C(S,t;K,T) - P(S,t;K,T) = \mathsf{e}^{-\int_t^T q(u)\mathsf{d}u} S - \mathsf{e}^{-\int_t^T r(u)\mathsf{d}u} K.$$

Nesting relationships with other models

- JDCEV model Nests standard time-homogeneous and inhomogeneous CEV models by appropriately setting parameters $r(t)$, $q(t)$, $a(t)$, b, c and β.

- JDCEV model Nests (when $b = 0, c < 1/2$) the *credit explosives* model proposed by Andreasen (2001), where the instantaneous credit spread is directly modeled by a diffusion process with quadratic drift and cubic instantaneous variance.

6 Asymptotic Analysis

Yield Spread

$$\mathcal{S}(S,T) = -\frac{1}{T}\ln B(S,0;T) - r = -\frac{1}{T}\ln Q(S,0;T),$$

where $Q(S,0;T)$ is the survival probability.

Short Maturity

$$\mathcal{S}(S,T) \sim b + c\,a^2 S^{2\beta} \quad \text{as} \quad T \to 0.$$

Long Maturity

Asymptotic Yield Spread

$$\mathcal{S}_\infty := \lim_{T\to\infty} \mathcal{S}(S,T).$$

$$\mathcal{S}_\infty = \begin{cases} b, & r - q + b \geq 0 \\ q - r, & r - q + b < 0 \end{cases}.$$

Probability of Ultimate Survival

$$\mathbb{Q}(\zeta = \infty | S_0 = S) = \lim_{T\to\infty} Q(S,0;T).$$

$$\mathbb{Q}(\zeta = \infty | S_0 = S) = \begin{cases} \left(\frac{x^2}{\tau_\infty}\right)^{\frac{1}{2|\beta|}} \mathcal{M}\left(-\frac{1}{2|\beta|}; \delta_+, \frac{x^2}{\tau_\infty}\right) > 0, & b = 0 \ \mathit{and} \ r - q > 0 \\ 0, & \mathit{otherwise} \end{cases},$$

where, for $b = 0$ *and* $r - q > 0$:

$$\tau_\infty = \lim_{T\to\infty} \tau(0,T) = \frac{a^2}{2|\beta|(r-q)},$$

7 Numerical Examples:

Jump to Default Extended CEV Model

Numerical Examples consider **σ(S, t) = σ(S), λ(S, t) = λ(S).** i.e. a(t) and b(t) are constants.

Stock Price Process before default (under the Risk-Neutral measure)

$$dS_t = [r(t) - q(t) + \lambda(S_t, t)]S_t\,dt + \sigma(S_t, t)S_t\,dB_t, \quad S_0 = S > 0, \tag{2.1}$$

where $r(t) \geq 0$, $q(t) \geq 0$, $\sigma(S, t) > 0$ and $\lambda(S, t) \geq 0$

Hazard Process

$$\Lambda_t = \int_0^t \lambda(S_u, u)du.$$

Volatility Process

$$\sigma(S, t) = a(t)S^\beta, \ \beta < 0 \tag{4.1}$$

$$\sigma(S) = \sigma_* \left(\frac{S}{S^*}\right)^\beta, \Big| \ \beta < 0$$

Intensity of the Jump Process

$$\lambda(S, t) = b(t) + c\,\sigma^2(S, t) = b(t) + c\,a^2(t)S^{2\beta}, \Big| \ \beta < 0 \tag{4.2}$$

$$\lambda(S) = b + c\,\sigma_*^2 \left(\frac{S}{S^*}\right)^{2\beta}. \Big| \ \beta < 0$$

In the Numerical Examples that follow,

S┌ = 50, β = −1, r = 0.05, q = 0. And b = 0, 0.02, σ∗ = 0.2, and c = 0, ½, 1.

7 Numerical Examples:

Default Probabilities in the JDCEV Model

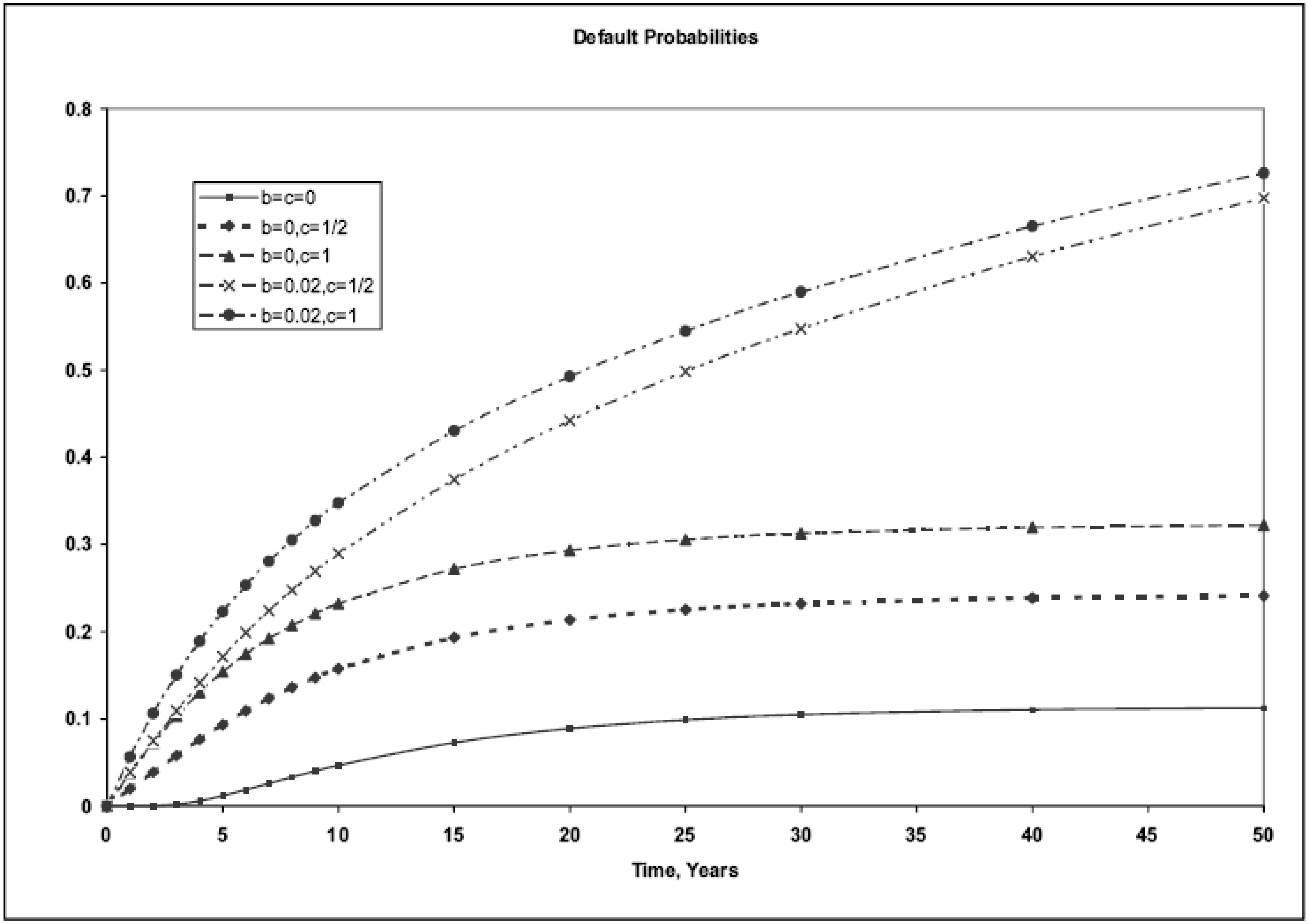


Figure 1: Risk-neutral default probabilities.
Parameter values: $S = S* = 50$, $\sigma* = 0.2$, $\beta = -1$, $r = 0.05$, $q = 0$, $b = 0, 0.02$, $c = 0, 1/2, 1$.

7 Numerical Examples:

Term Structure of the Credit Spreads in the JDCEV Model

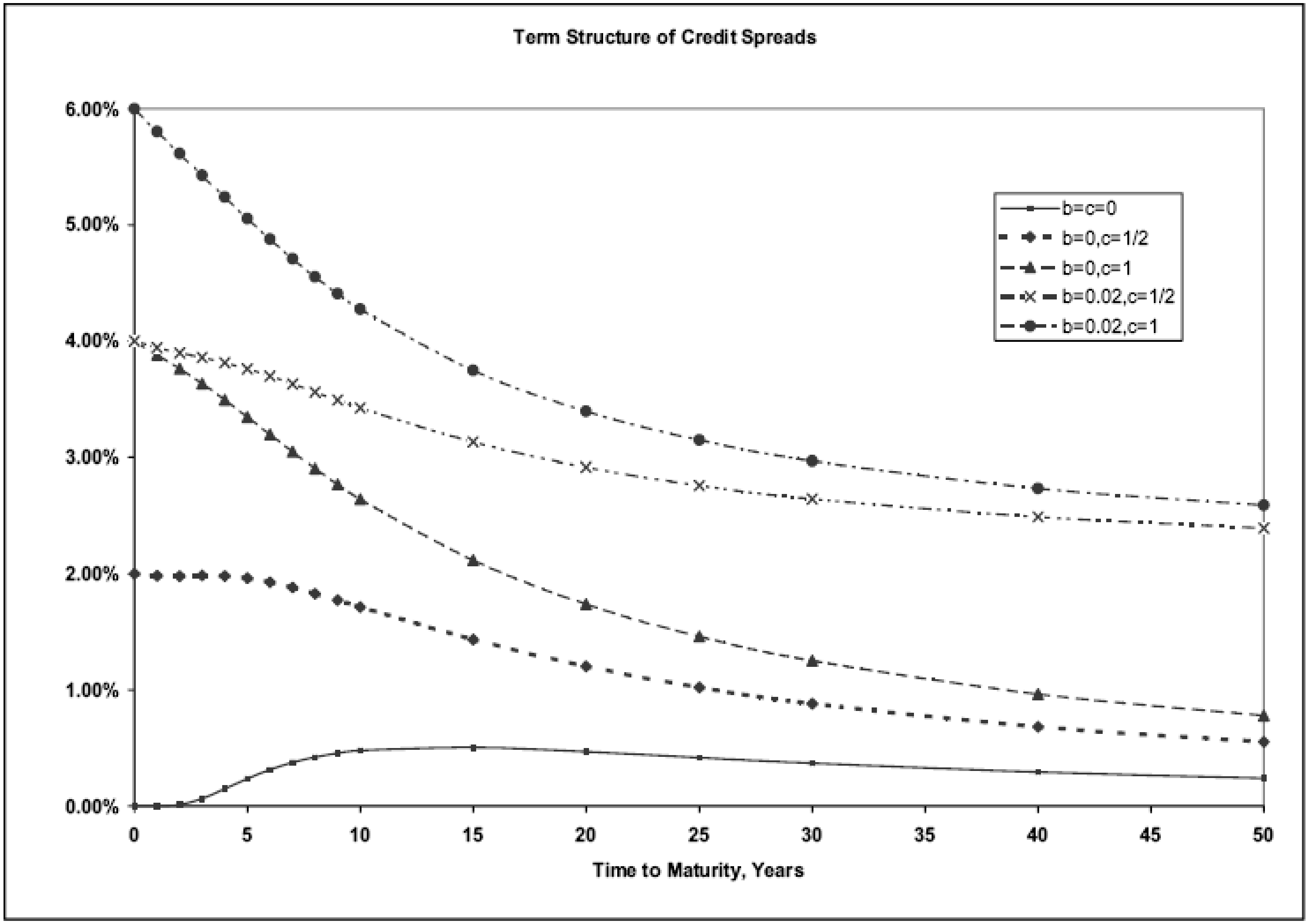


Figure 2: Term structures of credit spreads.
Parameter values: $S = S* = 50$, $\sigma* = 0.2$, $\beta = -1$, $r = 0.05$, $q = 0$, $b = 0, 0.02$, $c = 0, 1/2, 1$.

7 Numerical Examples:

Implied Volatility Skews in the JDCEV Model

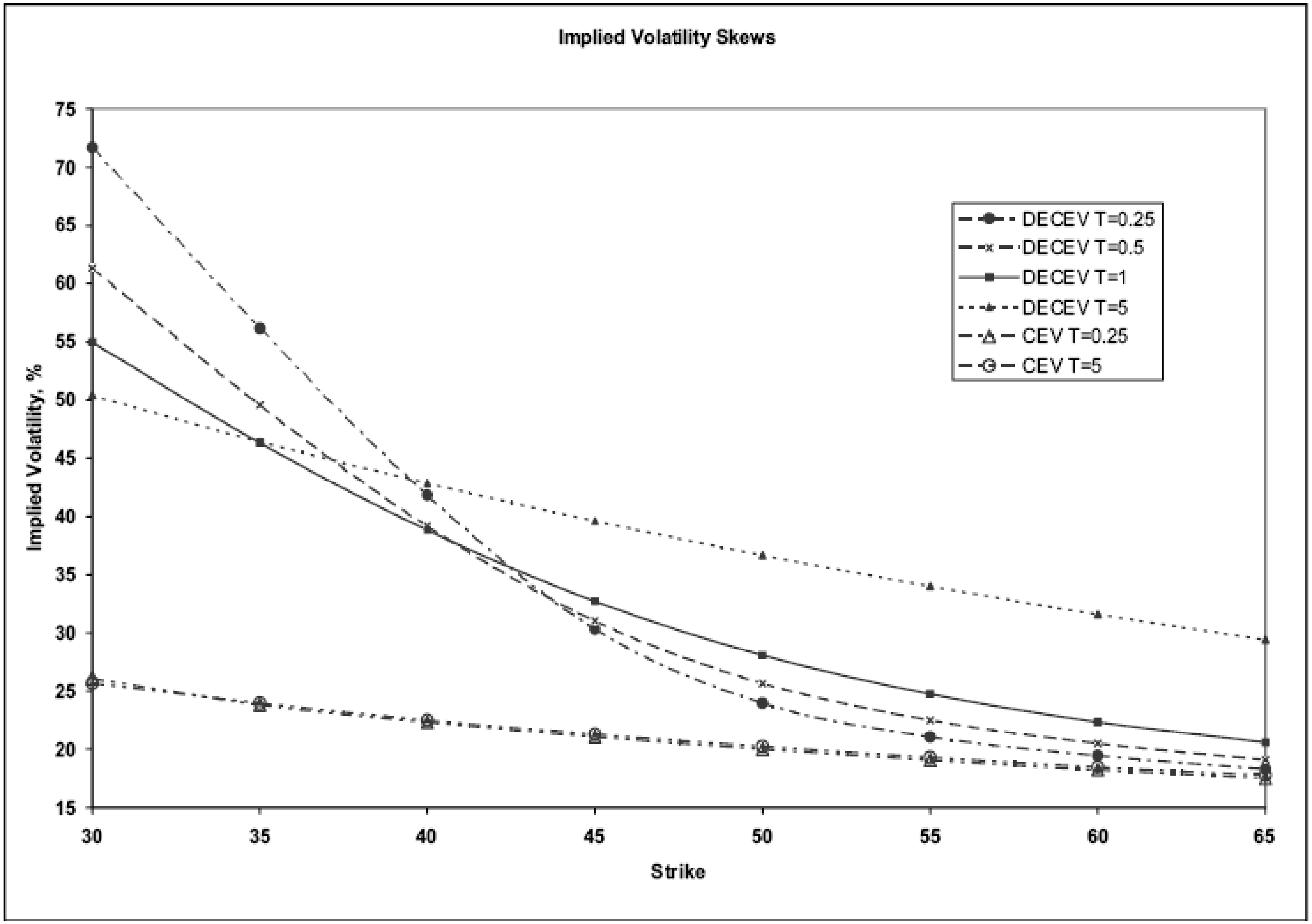


Figure 3: Implied volatility skews. Parameter values: $S = S^* = 50$, $\sigma^* = 0.2$, $\beta = -1$, $r = 0.05$, $q = 0$. For CEV model: $b = c = 0$. For JDCEV model: $b = 0.02$, $c = 1$. For JDCEV times to expiration are $T = 0.25, 0.5, 1, 5$ years. Implied volatilities are plotted against strike.

8 Put Options as Credit Derivatives:

The Default Claim Embedded in the Put Option

$$P(S,t;K,T) = P_0(S,t;K,T) + P_D(S,t;K,T)$$

$$= e^{-\int_t^T r(u)du}\mathbb{E}\left[e^{-\int_t^T \lambda(S_u,u)du}(K-S_T)^+ \mathbf{1}_{\{T_0>T\}}\middle| S_t = S\right]$$

$$+Ke^{-\int_t^T r(u)du}[1-Q(S,t;T)], \tag{3.8}$$

K	$P_0(S,t;K,T)$	$P_D(S,t;K,T)$	$P(S,t;K,T)$
5	3.3×10^{-8}	0.26819	0.26819
10	2.0×10^{-6}	0.53638	0.53638
20	0.00036	1.07277	1.07313
30	0.01499	1.60915	1.62414
40	0.23407	2.14553	2.37960
45	0.67715	2.41372	3.09087
50	1.62988	2.68192	4.31180
55	3.32780	2.95011	6.27791
60	5.88779	3.21830	9.10609
65	9.23827	3.48649	12.7248
70	13.1640	3.75468	16.9187
75	17.4224	4.02287	21.4453

Table 1: Put prices in JDCEV model (P = P0 + PD). Parameter values: S = S∗ = 50, σ∗ = 0.2, β = −1, r = 0.05, q = 0, b = 0.02, c = 1, T = 1 year.

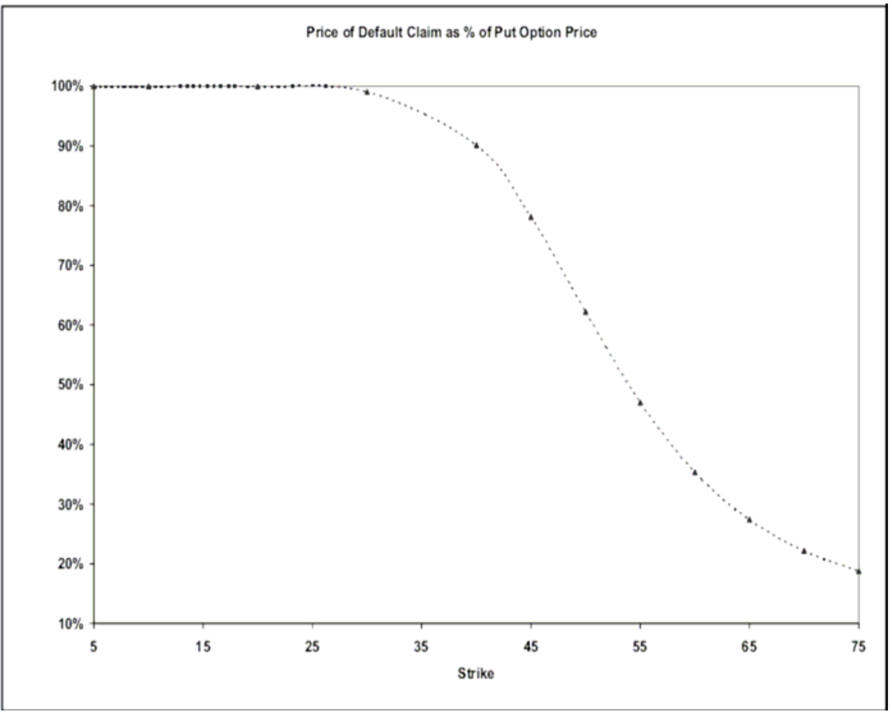


Figure 4: Price of default claim as percentage of the put option price as a function of the put strike. Parameter values: S = S∗ = 50, σ∗ = 0.2, β = −1, r = 0.05, q = 0, b = 0.02, c = 1, T = 1 year.

9 Summary

Objective:

A Flexible and Analytically Tractable Framework for Unified Valuation of

- Equity Derivatives – Stock Options
- Corporate Bonds and Credit Derivatives

Explain implied volatility skews, corporate yield spreads (asymptotic behavior), bond prices, prices of out of money puts

Modeling:

- Jump to Default Extended CEV (Constant Elasticity of Variance) Model for the pre-default stock dynamics under EMM

$$dS_t = [r(t) - q(t) + \lambda(S_t, t)]S_t\,dt + \sigma(S_t, t)S_t\,dB_t, \quad S_0 = S > 0,$$

- Hazard Process (with default as an absorbing state)

$$\Lambda_t = \int_0^t \lambda(S_u, u)du.$$

- Stock's realized volatility is negatively related to its price (leverage effect) and that implied volatilities are decreasing in the option's strike price (skew).

$$\sigma(S, t) = a(t)S^{\beta}, \ \beta < 0$$

- Default indicators such as credit default swap (CDS) spreads and corporate bond yields are positively related to historical volatility and implied volatilities of equity options.

$$\lambda(S, t) = b(t) + c\,\sigma^2(S, t) = b(t) + c\,a^2(t)S^{2\beta}, \ \beta < 0$$

9 Summary

Closed Form Solution:

- Deterministic changes of time and scale reduce the stock price process to a standard *Bessel process* with killing (Propositions 5.1-5.2, 5.3-5.4)
- Bessel process density is then expressed in terms of the *non-central chi-square density* (as on page 12 of the paper), and Lemma 5.1 is used to compute moments of the chi-squared distribution
- Then derive *explicit closed form solutions* (Proposition 5.5) for risk-neutral survival probabilities, CDS spreads, corporate bond values, and European-style equity options (using the framework discussed in Section 3)

Model Outputs:

- Asymptotic behavior of Credit Spreads is discussed
- Put Options' relationship with Credit Derivatives is studied
- Numerical Examples demonstrate the Default Probabilities, Credit Spreads and Implied Volatility Skews