Abstract. When granular material is modeled as a continuum, plastic constitutive behavior is often assumed. The use of plasticity amounts to replacing a complicated micromechanical system by its average behavior. Recent experiments have shown that, at least for small-scale systems, stress fluctuations may be of the same order, or even much larger, than average stresses.

In this paper a first generation of discrete models for stress fluctuations is discussed. These models consist of many spring-slider elements in parallel. The sliders all obey the same law for frictional resistance, and this resistance varies with the position, but not the velocity, of the slider. The initial positions (and hence the initial frictional resistances) of the sliders are taken to be random. The usual elastoplastic response emerges as the ensemble average over all possible initial positions of the sliders. The stress response resulting from any particular choice of initial conditions exhibits fluctuations similar to those in the experiments. It is shown that the magnitude of fluctuations is governed by two parameters, namely, the system size and the roughness, the latter defined as the ratio of particle contact length to particle size. In numerical simulations, it is observed that the roughness parameter controls the shape of the stress response as a function of applied strain.

Key words. plasticity, stress fluctuations, granular material

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1. Introduction. In experiments with sheared granular material, Miller, O’Hearn, and Behringer [12] observed that fluctuations in stress at a localized site on the boundary were comparable in magnitude to or even much larger than the average stress there. Admittedly, these experiments were performed on small systems: specifically, stresses were measured over an area in contact with only 20–50 grains. It is not yet known how large an area must be considered for stress fluctuations to become negligible compared to the average stress, or indeed whether this ever happens. However, it already seems clear from the existing data that, for small-scale granular flow such as that occurring, for example, in the pharmaceutical industry, accurate modeling requires consideration of stress fluctuations as well as their average values. In this paper we introduce and study a first-generation model for stress fluctuations.

When granular flow is modeled as a continuum, average stresses are usually determined from some form of an elastoplastic constitutive law. Our model for fluctuations resembles the Prager–Mroz discrete-element model for elastoplasticity (see [11, Example 5(b), p. 104] and [13, 14]). Let us recall the basics of their model, modified slightly to facilitate the transition to our model.

Consider simple shearing of a uniform block of elastoplastic material\footnote{One may think either of a pressure-insensitive material (e.g., a metal) or of a pressure-sensitive material (e.g., grain, soil, rock) under constant normal stress. We do not consider dilatency here.} under mono-
tonic, quasi-static loading (cf. Figure 1). Suppose that the force $F$ required to displace the upper surface of the block a distance $u$ has the form sketched in Figure 2; in words, the material exhibits hardening for $u < u_{\text{crit}}$ and perfect plasticity for $u > u_{\text{crit}}$. One discrete-element model for this behavior (cf. Figure 3) consists of an array of $N$ spring-slider links connecting two rigid rods. Because the deformation is quasi-static, the force $f_j$ in the $j$th spring equals the force transmitted to the lower rod by the $j$th slider. Let $m_j$ be the force needed to overcome the friction of the $j$th slider, what we shall call its frictional resistance. Thus,

$$f_j \leq m_j, \quad j = 1, \ldots, N,$$

and the slider is stationary unless equality holds in (1.1). Any one link has an elastic/perfectly plastic force-vs.-displacement curve, as sketched in Figure 4. By considering an array in which different sliders have different resistances $m_j$, one can arrange that the total force

$$F = \sum_{j=1}^{N} f_j$$

exhibits hardening for a range of $u$—say, $u \leq u_{\text{crit}}$—as in Figure 2. More accurately, the discrete-element system has a piecewise linear force-vs.-displacement curve, but if $N$ is large, one can obtain a close approximation of the smooth curve in Figure 2.

The sliders in the Prager–Mroz model move in a spatially uniform environment; i.e., the frictional resistance $m_j$ is independent of the slider’s position. By contrast, in our model for fluctuations, we assume that frictional resistance depends periodically on the slider’s position $x$, with the sawtooth form sketched in Figure 5a. Friction of this form could arise, for example, if a rough slider moves over a surface with alternate rough and smooth bands (cf. Figure 5b), provided we suppose that the

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2 $F$ is of course a tangential force. Although the experiments [12] concerned normal forces, in this paper we consider only shearing forces.

3 Since we consider only monotone loading, the forces $f_j$ are always nonnegative, and we may ignore lower bounds on $f_j$.

4 In drawing Figure 5, we assume that the sliders and the rough bands both have width $a$. If these widths were different, this would introduce extra parameters into the model, perhaps more than such a simple model can reasonably support.
Frictional resistance is proportional to the fraction of the area of the slider which overlaps a rough band on the surface.

To apply the model to sheared granular material, we shall suppose, very simplistically, that \( m(x) \) approximates the frictional resistance experienced by one grain as it slides over the layer of grains below it (cf. Figure 6). Thus the period \( \delta \) in Figure 5a may be interpreted as the grain diameter, and \( 2a \) may be interpreted as the interval of active contact between grains.

To synthesize the above: in this paper we study the force-vs.-displacement curve of an array of \( N \) spring-slider links connecting two rigid rods as in Figure 3, assuming that all springs are identical and that each slider experiences the same periodically varying frictional resistance \( m(x) \). We introduce statistical considerations by assuming that initial positions of the sliders are randomly distributed along the \( x \)-axis. The usual force-vs.-displacement curve of elastoplasticity will emerge as the ensemble average in our model, i.e., an average taken with respect to all possible initial positions for the sliders. But, as we shall see in section 4, the force-vs.-displacement curve for any particular choice of initial positions will exhibit vastly more structure than the smooth curve in Figure 2.

One striking consequence of our model deserves special mention. One might think that, in experiments, \( u_{\text{crit}} \) and \( F_{\text{crit}} \) in Figure 2 are totally unrelated parameters. However, in our model there is a dimensionless equation which connects these two parameters, and moreover the experimental data reported in [18] come remarkably close to satisfying this relation. (For more detail, see the remark immediately following Theorem 1.)

Spring-slider models for earthquakes were introduced by Burridge and Knopoff [4] and studied extensively by others (e.g., [1, 5, 6]). Despite some similarities to our model, there are also important differences. For one thing, the springs and sliders are connected differently in the two models. Most importantly, however, the earthquake models study large-magnitude, discontinuous, dynamical events (resulting from
Fig. 3. Discrete-element model for elastoplastic shearing.

Fig. 4. Force vs. displacement for one spring-slider.

Fig. 5. (a) The position-dependent force required to overcome friction.
kinetic friction being smaller than static friction), while we study quasi-static, continuous evolution. Incidentally, [8], in which the motion of sandpaper over a carpet is studied, provides another interesting point of comparison.

The remainder of this paper is organized as follows. In section 2 we derive equations which govern the evolution of the system we study; in section 3 we perform the ensemble average; and in section 4 we estimate the magnitude of stress fluctuations. Finally, in a rather chatty section 5, we make additional observations about this model and propose a variety of generalizations.

2. The basic mechanical system. In this section we (i) formulate equations for the evolution of the forces in the system we study and (ii) discuss units and issues regarding nondimensionalization of these equations.

2.1. Formulation of equations. Consider an array of $N$ springer-slider links connecting two rigid rods, as in Figure 3, where the frictional resistance $m(x)$ at each
slider depends on its position $x$ as in Figure 5a. In symbols, $m(x)$ is the extension of

$$
\phi(x) = \begin{cases} 
\frac{f_{\max}}{a_j} x, & 0 < x < a, \\
f_{\max} \left( 2 - \frac{x}{a} \right), & a < x < 2a, \\
0, & 2a < x < \delta,
\end{cases}
$$

to a $\delta$-periodic function. Note that three parameters, $f_{\max}, a$, and $\delta$, are contained in the definition of $m(x)$. Suppose that initially the force $f_j$ in each spring vanishes and the $j$th slider is located at $x^{(0)}_j$.

We seek relations to determine the forces $f_j$ and the slider positions $x_j$ that result from displacing the upper rod in Figure 3 quasi-statically a distance $u$ to the right. One such equation is

$$f_j = k[u - (x_j - x^{(0)}_j)]$$

since the elongation of the $j$th spring equals the displacement $u$ reduced by whatever amount the $j$th slider may have moved. The other relation contains an alternative depending on the magnitude of $u$. On the one hand,

$$f_j = \begin{cases} 
\frac{f_{\max}}{a_j} x, & 0 < x < a, \\
f_{\max} \left( 2 - \frac{x}{a} \right), & a < x < 2a, \\
0, & 2a < x < \delta,
\end{cases}
$$

because in this case the force in the $j$th spring generated by the motion of the upper rod will be insufficient to overcome the resistance to motion of its slider. On the other hand, we claim that

$$f_j = \begin{cases} 
\frac{f_{\max}}{a_j} x, & 0 < x < a, \\
f_{\max} \left( 2 - \frac{x}{a} \right), & a < x < 2a, \\
0, & 2a < x < \delta,
\end{cases}
$$

Certainly

$$f_j \leq m(x_j).$$

Thus (2.2b) expresses the condition that, once frictional resistance is overcome, the spring-slider link evolves so that it is always on the verge of sliding. This condition is intuitive if the system evolves continuously as $u$ is increased, i.e., if an infinitesimal increase in $u$ leads only to an infinitesimal increase in $x_j$. However, if a slider is located at a position $x_j$ where frictional resistance decreases (i.e., $m'(x_j) < 0$), then an infinitesimal increase in $u$ might cause a finite jump in $x_j$ to a new equilibrium for which the inequality (2.3) is strict, i.e., to an equilibrium not satisfying (2.2b)! In the following lemma we show that, provided

$$f_{\max} \frac{ka}{2} < 1,$$

the spring-slider links can evolve continuously. We shall assume (2.4) in the following, thereby justifying (2.2b). In section 5 we shall cite experimental data in partial support of (2.4).

\footnote{Note that $m(x_j^{(0)})$ might vanish; thus for some $j$ the first case may not be present.}

\footnote{Since we do not consider dynamics, we cannot discuss the stability of equilibria. Therefore, we can give only a plausibility argument for (2.2b).}

\footnote{If such jumps occur, the assumption of quasistatic deformation needs to be reexamined. The eventual equilibrium attained might depend on dynamic parameters not considered here.}
Lemma 2.1. If (2.4) is satisfied, then for any \( u, x^{(0)} \), the system
\[
\begin{align*}
  f &= k[u - (x - x^{(0)})], \\
  f &= m(x)
\end{align*}
\]
has a unique solution \( f, x \) which varies continuously with \( u \) and \( x^{(0)} \).

Proof. Eliminating \( f \) between the two equations, we obtain the single equation
\[
(2.5) \quad m(x) = k[u - (x - x^{(0)})]
\]
for \( x \) as a function of \( u \) and \( x^{(0)} \), which we analyze graphically. The right-hand side (RHS) of (2.5), considered as a function of \( x \), defines a line with slope \(-k\), while the left-hand side defines a curve whose slope satisfies
\[
|m'(x)| \leq \frac{f_{\text{max}}}{a}.
\]
If (2.4) holds, then the line is steeper than the maximum slope of the curve; thus there is a unique intersection. Continuity in \( u \) and \( x^{(0)} \) follows from similar consideration of the line and the curve. This completes the proof.

2.2. Units and parameters. The total force \( F \) resulting from displacement of the upper rod in Figure 3 by \( u \) is given by
\[
(2.6) \quad F = \sum_{j=1}^{N} f_j,
\]
where \( f_j \) is determined from solving (2.1), (2.2). To interpret the predictions of the model, it is helpful to introduce stress and strain in place of force and displacement. Thus we define
\[
(2.7) \quad \begin{align*}
  (a) & \quad \Sigma = \frac{F}{N\delta^2}, \quad (b) \quad \epsilon = \frac{u}{\delta}
\end{align*}
\]
where \( N \) is the number of spring-slider links and \( \delta \) is the period of \( m(x) \) (cf. Figure 5a). These scalings come from regarding the system of Figure 3 as a crude model for the forces generated by one layer of grains in a granular material sliding over the layer below it, where the grain diameter is of the order \( \delta \). Thus, regarding stresses, each grain occupies an area of the order of \( \delta^2 \), so (2.7a) represents the total force divided by the area occupied by \( N \) grains. Similarly, regarding strains, the separation between layers in Figure 3 is of the order of the grain diameter \( \delta \), so (2.7b) represents the strain.\(^8\)

Following the procedure of [10, Chapter 6], let us determine the dimensionless parameter groups in the equations (2.1), (2.2). These equations contain the five parameters in Table 1,\(^2\) in the listing of their units, \( M, L \), and \( T \) denote mass, length, and time respectively. To find dimensionless groups, we consider combinations
\[
N^{p_1} k^{p_2} f_{\text{max}}^{p_3} \delta^{p_4} a^{p_5}
\]

\(^8\)Note that one may decompose \( \epsilon \) into a plastic strain
\[
\epsilon_{\text{pl}} = \frac{1}{N\delta} \sum_{j=1}^{N} (x_j - x_j^{(0)})
\]
and an elastic strain \( \epsilon_{\text{el}} = \epsilon - \epsilon_{\text{pl}} \).

\(^9\)We do not include the initial data \( x_j^{(0)} \) in this list of parameters because, starting in the next section, we will be averaging over all initial data.
for all exponents $p_i$. We find that there are three independent dimensionless parameters, which we choose to be

$$\eta \equiv \frac{f_{\text{max}}}{k\delta}, \quad \text{and} \quad \alpha \equiv \frac{a}{\delta}.$$  

The parameter $\eta$ determines the balance between frictional and elastic forces, while $\alpha$ equals the ratio of the active-contact length to grain diameter. Throughout this paper, $\eta$ and $\alpha$ will be defined by (2.8).

Note that, in terms of these parameters, (2.4) requires that

$$\frac{\eta}{\alpha} < 1.$$  

We also observe from Figure 5a that

$$\alpha < \frac{1}{2}.$$  

Although we could nondimensionalize all variables in (2.1), (2.2), we do not find it convenient to do so.

3. Average behavior.

3.1. Formulation of the main result. In posing random initial conditions for (2.1), (2.2), we want to assume that $x_j^{(0)}$ satisfies

$$j\delta \leq x_j^{(0)} < (j + 1)\delta,$$

with all positions in this interval being equally likely. More precisely, let us write

$$x_j^{(0)} = j\delta + \gamma_j;$$

we consider an ensemble of systems (2.1), (2.2) indexed by $\gamma = (\gamma_1, \ldots, \gamma_N)$, with $\gamma$ uniformly distributed over $[0, \delta]^N$. In probabilistic terms, $\gamma_1, \ldots, \gamma_N$ are independent random variables, uniformly distributed over $[0, \delta]$.

It follows from Lemma 2.1 that, for any initial data, (2.1), (2.2) determine $f_j$ uniquely as a continuous function of $u, x_j^{(0)}$. Let us write

$$\Sigma(\varepsilon, \gamma_1, \ldots, \gamma_N)$$

for the stress obtained by choosing initial data according to (3.1), solving (2.1), (2.2) for $f_j$, computing $F$ from (2.6), and normalizing according to (2.7). In the following theorem, the central result of section 3, we compute the ensemble average, or expected value, of $\Sigma(\varepsilon, \gamma)$, i.e.,

$$E[\Sigma(\varepsilon, \cdot)] = \frac{1}{\delta^N} \int_0^\delta d\gamma_1 \cdots \int_0^\delta d\gamma_N \Sigma(\varepsilon, \gamma_1, \ldots, \gamma_N).$$
The theorem establishes a critical strain \( \varepsilon_{\text{crit}} \) with the property that \( E[\Sigma(\varepsilon, \cdot)] \) exhibits hardening for \( \varepsilon < \varepsilon_{\text{crit}} \) and perfect plasticity for \( \varepsilon > \varepsilon_{\text{crit}} \), as shown in Figure 2.

**THEOREM 1.** For \( \varepsilon < \varepsilon_{\text{crit}} \)

\[
E[\Sigma(\varepsilon, \cdot)] = G \left( 1 - \frac{\varepsilon}{2\varepsilon_{\text{crit}}} \right) \varepsilon,
\]

and for \( \varepsilon > \varepsilon_{\text{crit}} \)

\[
E[\Sigma(\varepsilon, \cdot)] = \Sigma_{\text{crit}},
\]

where \( \varepsilon_{\text{crit}} \), the elastic modulus \( G \), and the ultimate yield stress \( \Sigma_{\text{crit}} \) are given by

\[
\begin{align*}
(a) & \quad \varepsilon_{\text{crit}} = \eta, \\
(b) & \quad G = 2 \alpha \frac{k}{\delta}, \\
(c) & \quad \Sigma_{\text{crit}} = \alpha \frac{f_{\text{max}}}{\delta^2} = \frac{\eta}{2} G.
\end{align*}
\]

**Remark.** In the experiments of Vardoulakis and Graf [18]\(^{10}\)

\[ \varepsilon_{\text{crit}} \approx 0.05 \quad \text{and} \quad \frac{\Sigma_{\text{crit}}}{G} \approx 0.02. \]

According to (3.5), in our model these two, apparently independent, dimensionless numbers are both determined by the single parameter \( \eta \). Moreover, the agreement with (3.6) is remarkably good, for such a simple model, if we take \( \eta \) in the range

\[ 0.04 \leq \eta \leq 0.05. \]

To prove the theorem, first we find an explicit formula for the solution of (2.1), (2.2), and then we compute the average of the sum of the individual forces.

**3.2. Solution of (2.1), (2.2).** Let us rewrite (2.1), (2.2), temporarily suppressing the index \( j \) and defining \( s = x - x^{(0)} \) as the amount of slippage; we obtain

\[
f = k(u - s),
\]

\[
\begin{align*}
\text{(a)} & \quad \text{if } ku < m(\gamma), \text{ then } s = 0, \\
\text{(b)} & \quad \text{if } ku > m(\gamma), \text{ then } f = m(\gamma + s).
\end{align*}
\]

In (3.9), we have used the fact that \( m(x) \) is \( \delta \)-periodic to conclude from (3.1) that \( m(x^{(0)}) = m(\gamma) \). Note that if

\[ ku > f_{\text{max}}, \]

then the alternative (3.9a) will never obtain. This inequality may be reexpressed in terms of strain as

\[ \varepsilon = \frac{u}{\delta} > \frac{f_{\text{max}}}{k\delta} = \eta. \]

\(^{10}\)We do not mention the value of \( G \) in the experiments since of course this dimensional parameter could be matched exactly by an appropriate choice of \( k/\delta \).
Thus if we define $\varepsilon_{\text{crit}}$ by (3.5a), we see that different behavior may be expected from (3.8), (3.9) if $\varepsilon < \varepsilon_{\text{crit}}$ or $\varepsilon > \varepsilon_{\text{crit}}$.

In the simpler case, $\varepsilon > \varepsilon_{\text{crit}}$, the force $f$ satisfying (3.8), (3.9b) depends on the two arguments $u$ and $\gamma$ only through their sum:

$$f = \Psi(u + \gamma).$$

(3.10)

This form follows from the fact that (3.8), (3.9b) are unchanged by the substitution

$$u \mapsto u + c, \quad \gamma \mapsto \gamma - c, \quad s \mapsto s + c$$

for any real number $c$. We claim that $\Psi$ is the piecewise linear, periodic function\(^{11}\) illustrated in Figure 7. To see this, note that $m(x)$ is piecewise linear, with the three distinct intervals of linearity per period

$$(3.11) \quad (0, a), \quad (a, 2a), \quad (2a, \delta).$$

For $s + \gamma$ in one of these intervals, (3.8), (3.9b) reduce to a linear system for $f$ and $s$. The three intervals (3.11) give rise to the three intervals in $\Psi$

$$\left(0, a \left(1 + \frac{\eta}{\alpha}\right) \right), \quad \left(a \left(1 + \frac{\eta}{\alpha}\right), 2a \right), \quad (2a, \delta).$$

Solving for $f$ and $s$ in each interval gives the formula for $f = \Psi(u + \gamma)$ shown graphically in Figure 7.

If $\varepsilon < \varepsilon_{\text{crit}}$, a fourth case, beyond (3.11), arises in solving (3.8), (3.9), i.e., the case where the alternative (3.9a) obtains. As a result, the solution $f$ depends essentially on both $u$ and $\gamma$. For comparison with (3.10) it is convenient to write the solution in the form

$$f = \Phi(u + \gamma, u).$$

(3.12)

Solving for $f$ and $s$ as above, we see that $\Phi(\cdot, u)$ is obtained from $\Psi(\cdot)$ by “cutting off” the peaks of $\Psi$ at the height $ku$, as illustrated in Figure 8. Of course $ku < f_{\text{max}}$ precisely when $\varepsilon < \varepsilon_{\text{crit}}$.

\(^{11}\)Note by (2.9) that the graph of $\Psi$ makes sense, i.e., $\left(1 + \frac{\eta}{\alpha}\right) a < 2a$.  

Fig. 7. A graph of $\Psi$ in (3.10).
3.3. Taking the expected value. By (3.10), if $\varepsilon > \varepsilon_{\text{crit}}$,

$$(3.13) \quad F(u, \gamma_1, \ldots, \gamma_N) = \sum_{j=1}^{N} \Psi(u + \gamma_j).$$

Therefore

$$E[F(u, \cdot)] = \frac{N}{\delta} \int_{0}^{\delta} \Psi(u + \gamma) d\gamma = \frac{N}{\delta} af_{\text{max}}.$$

Recalling the scaling (2.7), we verify (3.4) and the first equality in (3.5c). Similarly, if $\varepsilon < \varepsilon_{\text{crit}}$,

$$E[F(u, \cdot)] = \frac{N}{\delta} \int_{0}^{\delta} \Phi(u + \gamma, u) d\gamma.$$

The integral is performed easily with the aid of Figure 8. The formulas (3.3), (3.5b), and the second equality in (3.5c) now follow from this calculation and (2.7). The proof of Theorem 3.1 is complete.

4. Fluctuations in the perfectly plastic regime. Let us contrast the behavior of $\Sigma(\varepsilon, \gamma)$ for one specific, randomly sampled $\gamma \in [0, \delta)^N$ with that of the ensemble average $E[\Sigma(\varepsilon, \cdot)]$. According to (3.13) and (2.7), when $\varepsilon > \varepsilon_{\text{crit}}$

$$(4.1) \quad \Sigma(\varepsilon, \gamma) = \frac{1}{N \delta^2} \sum_{j=1}^{N} \Psi(\delta \varepsilon + \gamma_j),$$

and when $\varepsilon < \varepsilon_{\text{crit}}, \Sigma(\varepsilon, \gamma)$ has a similar representation in terms of $\Phi$. Since $\Psi$ and $\Phi$ are piecewise linear, nonmonotone functions, it follows that $\Sigma(\varepsilon, \gamma)$ is a piecewise linear, nonmonotone function of $\varepsilon$—very different from the smooth, monotone behavior of $E[\Sigma(\varepsilon, \cdot)]$ illustrated in Figure 2. Moreover, since $\Psi$ is periodic with period $\delta$, the stress $\Sigma(\varepsilon, \gamma)$ is periodic in $\varepsilon$ with period 1 when $\varepsilon > \varepsilon_{\text{crit}}$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig8}
\caption{A graph of $\Phi$ in (3.12).}
\end{figure}
In Figure 9 we show numerical simulations of $\Sigma(\varepsilon, \gamma)$ for several $\gamma$'s chosen using a random-number generator, with various values for the nondimensional parameters $N, \eta, \alpha$. In most of the graphs, we have chosen $\eta/\alpha$ close to its limiting value of $\eta/\alpha = 1$, imposed by (2.9). We also show a simulation with a larger value of $\alpha$, illustrating a less noisy response in the time series when $\eta/\alpha$ is far away from its limit. Note that the magnitude of fluctuations in this simulation is decreased only slightly (by a factor of $\sqrt{2}$ approximately, according to Theorem 2 below), whereas the time series is dramatically different. We note from the simulations that

(i) fluctuations are substantial, and their magnitudes depend on the parameters;
(ii) even with identical parameters, the stress histories for different choices of $\gamma$ may appear rather different.

Point (ii) concerns the analysis of $\Sigma(\varepsilon, \gamma)$ as a time series, which we shall pursue further in [16] in the context of a slightly more sophisticated model (cf. item (c), section 5.2).
Regarding point (i), in the following theorem we estimate the magnitude of fluctuations by computing the variance of a random variable: specifically, of \( \Sigma(\varepsilon, \gamma) \), which for fixed \( \varepsilon \) is a random variable by virtue of being a function of the \( N \) independent random variables \( \gamma_1, \ldots, \gamma_N \). To help with interpretation in (4.3) below, we normalize the variance by \( \Sigma_{\text{crit}} \), the expected value of \( \Sigma(\varepsilon, \gamma) \). As the theorem shows, \( \text{Var}(\Sigma(\varepsilon, \cdot)) \) is in fact independent of \( \varepsilon \) for \( \varepsilon > \varepsilon_{\text{crit}} \).

**Theorem 2.** If \( \varepsilon > \varepsilon_{\text{crit}} \),

\[
\frac{\text{Var}(\Sigma(\varepsilon, \cdot))}{\Sigma_{\text{crit}}} = \frac{1}{\sqrt{N}} \left\{ \frac{2 - 3\alpha}{3\alpha} \right\}^{1/2}.
\]

**Proof.** The proof is based on (4.1). Since \( \gamma_1, \ldots, \gamma_N \) are independent and identically distributed, so are \( \Gamma_1, \ldots, \Gamma_N \), where

\[\Gamma_j = \Psi(\delta \varepsilon + \gamma_j).\]

In particular,

\[
\text{Var}(\Sigma(\varepsilon, \cdot)) = \frac{1}{N^2 \delta^4} \sum_{j=1}^{N} \text{Var}(\Gamma_j).
\]

We may compute from knowledge of \( \Psi \) (cf. Figure 7) that each \( \Gamma_j \) has a mixed discrete/continuous distribution. Specifically, \( \Gamma_j = 0 \) with probability \( 1 - \frac{2\alpha}{\delta} \); otherwise \( \Gamma_j \) is uniformly distributed in the interval \((0, f_{\text{max}})\). In symbols, \( \Gamma_j \) has the probability density

\[
\rho(x) = (1 - 2\alpha)\delta(x) + \frac{2\alpha}{f_{\text{max}}} \chi_{[0, f_{\text{max}}]}(x),
\]

where \( \delta(x) \) is the Dirac delta function and \( \chi_I \) is the characteristic function for an interval \( I \). Calculating the variance from the density \( \rho \), we find

\[
\text{Var}(\Gamma_j) = E[\Gamma_j^2] - (E[\Gamma_j])^2 = \frac{\alpha(2 - 3\alpha)}{3} f_{\text{max}}^2.
\]

We complete the derivation of (4.3) by substituting (4.5) into (4.4) and recalling (3.5c).

**Remark.** By the central limit theorem, if \( N \) is large, then \( \Sigma(\varepsilon, \gamma) \) has approximately a normal distribution.
5. Concluding remarks.

5.1. Observations on the present model. (a) Regarding the interpretation of dimensionless parameters, we argue from (3.5) that \( \eta \) determines average behavior in the model. Indeed (3.5a), (3.5c) involve only \( \eta \) and the dimensional parameter \( G \); if \( \eta \) is held fixed while \( N \) and \( \alpha \) are changed arbitrarily, we may adjust the dimensional parameter \( k/\delta \) in (3.5b), which merely defines the stress scale, so that \( G \) is unchanged.

We see from Theorem 2 that \( N \) and \( \alpha \) determine the magnitude of fluctuations. It is hardly surprising that the magnitude of fluctuations depends on \( N \), the system size. It is noteworthy, however, that even for fixed system size \( N \) and fixed average behavior (i.e., \( \eta \)), the size of fluctuations can be changed by varying \( \alpha \). Perhaps \( \alpha \) provides a realization of the “roughness parameter” on which the existence and amplitude of porosity waves in hopper flow was observed to depend [3, 2].

(b) With assumption (2.9), which is equivalent to (2.4), we have ruled out the necessity of discontinuous jumps in the evolution of the system as \( u \) is increased. The small size of \( \eta \) in (3.7) supports this assumption: If grains in Figure 6 make active contact over at least one tenth of a grain diameter, then \( 2a/\delta \geq 0.1 \), so \( \alpha \geq 0.05 \), and hence (2.9) is satisfied.

On the other hand, a \( 1/f^2 \) power spectrum at high frequencies was observed in the time series for the stress [12]. This result suggests that jumps may be contained in the experimental data, since the Fourier transform of a function with jump discontinuities decays like \( 1/f \), producing a \( 1/f^2 \) power spectrum.

(c) In preparing this paper, we have considered various forms for the frictional resistance \( m(x) \) in addition to the one illustrated in Figure 5a. In general terms, although precise formulas are changed by using a different \( m(x) \), the qualitative behavior of the model is not changed very much.

5.2. Straightforward generalizations of the model. (a) The Prager–Mroz model sheds useful insight on material response in elastoplasticity when unloading follows loading. It would be interesting to consider the present model under non-monotone loading.

(b) If (2.4) is not satisfied, then the evolution of the system will include jumps. Dynamic effects would need to be incorporated into the model to study these jumps. One wonders whether, as in other models [4, 5], an infinitesimal change in \( u \) could lead to a “big event,” i.e., an event during which the position of many sliders change by a finite amount.

(c) In [16] we analyze a generalization of the present model which decreases its rigid periodicity. Specifically, we assume that the \( j \)th slider experiences a frictional resistance

\[
m_j(x) = \sum_{k=1}^{\infty} \phi(x - \gamma_k^{(j)}),
\]

where

\[
\phi(x) = \begin{cases} 
\frac{f_{\text{max}}}{a} x, & 0 < x < a, \\
\frac{f_{\text{max}}}{a} \left( 2 - \frac{x}{a} \right), & a < x < 2a, \\
0, & \text{otherwise}
\end{cases}
\]

\[\text{In item (e) below, we remark that jumps may occur in a multilayer model of our type even if (2.4) is satisfied.}\]
and the points \( \gamma_k \), \( k = 1, 2, \ldots \), are the arrival times of a Poisson process [15] with mean separation \( \delta \). We prove that time and ensemble averages are equal, and we compute (temporal) correlations and power spectra.

(d) The frictional-resistance function \( m(x) \) in Figure 5a contains three parameters: \( f_{max}, a, \) and \( \delta \). In the previous generalization, we are in effect introducing randomness in \( \delta \). It is natural to also introduce randomness in \( f_{max} \) and \( a \). In this connection, let us mention the observation of van der Ziel [17] that, by superimposing pulses occurring at random times and having random decay rates, one may obtain a signal whose power spectrum has \( 1/f \) behavior over an arbitrarily large frequency range. Incidentally, both the experimental results [12] and the model [6] suggest that \( f_{max} \) ought to have a \( \Gamma \)-distribution, i.e., the probability density of \( f_{max} \) ought to be proportional to

\[
(f_{max})^p \exp\{-f_{max}/A\}
\]

for some power \( p \) and force scale \( A \).

(e) One of the many extreme simplifications of our model is that it concerns just one layer of particles in a granular medium. As a slightly more realistic model of granular material, we plan to study a matrix of spring-slider systems connected by rigid rods as illustrated in Figure 10, assuming the topmost rod is displaced quasistatically while the bottom rod is held fixed. Preliminary research has indicated that new phenomena will occur. In particular, it appears that a many-layer system may suffer jumps even while each individual spring/slider satisfies (2.4).

5.3. A challenging generalization of the model. The presumed cause of local-stress fluctuations in the experiments [12] is the formation and dissolution of stress chains. (See [9] for direct observations of stress chains using photoelasticity.) Of course stress chains cannot be captured in our model in which the “grains” are only one layer deep. Nevertheless, it is quite possible that the triangular peaks, of which \( m(x) \) in Figure 5a is composed, afford an adequate phenomenological representation of the forces exerted on the vessel walls by stress chains, provided \( f_{max}, a, \) and \( \delta \) are chosen randomly. (The appropriate probability distributions for \( f_{max}, a, \) and \( \delta \) would relate to the behavior of stress chains rather than of individual grains.) Indeed, it appears that the data of [12], which do not include information about spatial correlations at neighboring sites, can be fit rather well with a model of this type.

In this paper we have assumed independence (i.e., no correlation) between neighboring sites.\(^{13}\) Experiments so far do not provide significant evidence regarding spatial correlations. However, even if local stress at neighboring sites on the boundary do turn out to be correlated, models assuming independence may provide an upper bound for stress fluctuations in the real problem, for the following reason: If a correlation exists, we expect it to be a negative correlation since, if a large-stress chain ends at one site on the boundary, this would seem to reduce the probability that another large-stress chain would end at a nearby site. Of course, if \( X_1 \) and \( X_2 \) are negatively correlated random variables, then

\[
\text{Var}(X_1 + X_2) < \text{Var}(X_1) + \text{Var}(X_2).
\]

Thus models assuming independence may be useful for engineering design, where one must be conservative.

Despite the above considerations, which suggest that models with independence may have a continuing role to play, one would like to derive the behavior of stress

\(^{13}\)Indeed, it was the assumption of independence that made an analytical solution so accessible.
chains, in particular their spatial correlations, from first principles. Reference [6] represents a noteworthy first attempt in that direction.\textsuperscript{14}

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REFERENCES


\textsuperscript{14}Incidentally, this model predicts at least approximate, and sometimes exact, independence.


