

**USING LINEAR
PROGRAMMING TO
CONSTRUCT MARKOV
MOVES IN
CONTINGENCY TABLES**

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Abstract: A *partially specified contingency table* is the set of all contingency tables satisfying fixed marginal totals (minimal sufficient statistics). We present theory and methods for drawing a random sample of tables from this set based on linear programming. Diaconis and Sturmfels (1998) presented the original method which is based on uniform random sampling from a Markov basis for the set of all feasible moves between tables. The Markov basis is obtained from a corresponding Gröebner basis, which consequently must be available prior to sampling. Our approach is to construct random moves one at a time based on a linear programming formulation that assures irreducibility and reversibility of the Markov chain. Consequently, the set of all moves constructible by our procedure is a Markov basis. The method has the following features: it does not require a Gröebner basis; it is not affected by complexities of table design or unduly limited by table size; and, by virtue of an additional linear constraint specific to the current solution, it does not create infeasible moves that are immediately discarded. We illustrate related points through specific examples.

MARKOV BASIS, LINEAR PROGRAMS AND CONTINGENCY TABLES

Contingency Tables

- $AX = b$, A is $\{-1, 0, +1\}$, b is integer
- Current integer solution: $m^{(0)} \in \mathbb{N}^d : Am^{(0)} = b$
- Continuous Kernel: $Ker(A) = \{x \in \mathbb{R}^d : Ax = 0\}$
- Integer Kernel: $Ker_Z(A) = \{n \in \mathbb{Z}^d : An = 0\}$
- $Ker_Z(A, m^{(0)}) = \{n \in Ker_Z(A) : n + m^{(0)} \geq 0\}$
- *Table of Network Type*
 - o A totally unimodular
 - o all polyhedral extreme points are integer
 - o can solve MILP as network LP

Kernel LP

- continuous kernel restricted to unit hypercube

$$\min c(x^+ - x^-)$$

$$A(x^+ - x^-) = 0$$

$$LP(A, m_0): 0 \leq x^+, x^- \leq 1$$

$$m_0 + x^+ - x^- \geq 0$$

SOME FACTS

$$x = (x^+ - x^-) \in \text{Ker}(A)$$

$(x^+, x^-) \in LP(A, m^{(0)}) = \text{convex comb. of extreme points}$

$$(e^+, e^-)$$

$(x^+, x^-) \in F = \text{boundary of } LP(A, m^{(0)}) \rightarrow \exists j; * = -, + : x_j^* = 1$

$n \in \text{Ker}_Z(A) : n = tx, t \in Z^+, x \in F$

(e^+, e^-) extreme point $\rightarrow e_j^+ e_j^- = 0$ (*irredundant*)

$$\text{Supp}((x^+, x^-)) = \{j : x_j^+ + x_j^- > 0\}$$

$$\text{Supp}(x) = \text{Supp}((x^+ - x^-)) = \{j : x_j^+ - x_j^- \neq 0\}$$

Suffices to consider only *irredundant* points

$$(x^+, x^-), (e^+, e^-), (n^+, n^-)$$

$\text{Val}(n) = \text{Val}(n^+ - n^-) = \sum_j (n_j^+ + n_j^-)$ is well-defined

Move $n \in \text{Ker}_Z(A, m^{(0)})$ is **minimal** if

- 1) it cannot be expressed as the sum $n = n^{(1)} + n^{(2)}$ of two moves with strictly smaller support
- 2) if $\text{Supp}(n') = \text{Supp}(n)$ then $\text{Val}(n') \geq \text{Val}(n)$

If n minimal and $\text{Supp}(n') = \text{Supp}(n)$ then $n' = tn, t \in Z$

Minimal moves are sufficient for a Markov basis

Tables of Network Type (C 2007)

- kernel LP is network
- all extreme points are $\{0, 1\}$
- LP extreme points $(e^+, e^-) \leftrightarrow$ square-free
Markov moves $n = e^+ - e^-$
- stochastic LP (network) algorithm for selecting local moves resulting in a random sample of contingency tables subject to fixed marginals (MSS), as in (DS 1998)
- many (familiar) tables are network
 - o 2-D tables and hierarchies
 - o thin tables
 - o tables defined by 2 sets of MSS

Non-Network Tables

- some extreme points yield square-free minimal Markov moves
- the others, after clearing denominators, yield other Markov moves

EXAMPLES

Example 1 3x3x2, fixed 2-D marginals (MSS)
(no 3-way interaction model)
Thin table: Network (C 2003, 2007)

$$m^{(0)} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Kernel polyhedron extreme point (e^+ , e^-)

$$e^+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$e^- = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Kernel element

$$e^+ - e^- = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

Gröebner Basis element

$$n: x_{131}x_{221}x_{311}x_{112}x_{232}x_{322} - x_{111}x_{231}x_{321}x_{132}x_{222}x_{312}$$

Square-free (minimal) Markov move in $\text{Ker}_Z(A, m^{(0)})$

$$n = e^+ - e^- = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

Alternative table (solution)

$$m^{(1)} = m^{(0)} + n = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Example 2 3x3x3 table, fixed 2-D marginals (MSS)
 (no 3-way interaction model)
 Non-Network (C 2003)

$$m^{(0)} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Kernel polyhedron extreme point (e^+ , e^-)

$$e^+ = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 0 \\ 1/2 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \quad \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{pmatrix}$$

$$e^- = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1/2 \\ 0 & 0 & 0 \\ 1/2 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$$

Kernel element $e^+ - e^- =$

$$\begin{pmatrix} -1 & 1/2 & 1/2 \\ 1/2 & 0 & -1/2 \\ 1/2 & -1/2 & 0 \end{pmatrix} \begin{pmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 0 & 1/2 \\ 0 & 1/2 & -1/2 \end{pmatrix}$$

Gröebner Basis element

$$n: x_{121}x_{131}x_{211}x_{311}x_{112}x_{332}x_{113}x_{323} - x_{111}^2x_{231}x_{321}x_{132}x_{312}x_{123}x_{213}x_{333}$$

(Minimal) Markov move in $\text{Ker}_Z(A, m^{(0)})$

$$n = 2(e^+ - e^-) = \begin{pmatrix} -2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

Alternative table (solution)

$$m^{(1)} = m^{(0)} + n = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Note:

$$\begin{aligned} & \begin{pmatrix} -2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \\ = & \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix} + \\ & \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \\ & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} \end{aligned}$$

but this sequence is not locally feasible

Example 3 3x3 table with subtotal
Non-network (HTY 2008, CG 1989)

$$m^{(0)} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$$

subject to

- row totals
- column totals
- $x_{11} + x_{22} = 2$

“Half table”

$$(1/2)m^{(0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Non-integer “half table” polyhedron extreme point

$$\begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So, tables with subtotals are non-network

Kernel element

$$e^+ - e^- = \begin{pmatrix} -1/2 & 1/2 & 0 \\ 1/2 & 1/2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

Gröebner Basis element

$$n : x_{12}x_{21}x_{22}x_{33}^2 - x_{11}x_{23}^2x_{32}^2$$

(Minimal) Markov move in $\text{Ker}_Z(A, m^{(0)})$

$$n = 2(e^+ - e^-) = \begin{pmatrix} -2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

Alternative table (solution)

$$m^{(1)} = m^{(0)} + n = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Examples 4 2x2x2x2 table, all 2-D marginals (MSS)

4(a)

$$m = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

4(b)

$$m = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Non-integer extreme points

$$\begin{pmatrix} 1 & 1/2 \\ 1/2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \quad \begin{pmatrix} 5/3 & 0 \\ 0 & 1/3 \end{pmatrix} \begin{pmatrix} 0 & 1/3 \\ 1/3 & 1/3 \end{pmatrix} \\ \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \quad \begin{pmatrix} 0 & 1/3 \\ 1/3 & 1/3 \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & 0 \end{pmatrix}$$

So, non-network

Full tables

$$m^{(0)} = 2m = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

$$m^{(0)} = 3m = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}$$

Kernel elements $e^+ - e^-$

$$\begin{array}{cc}
 \mathbf{4(a)} & \mathbf{4(b)} \\
 \begin{pmatrix} 1 & -1/2 \\ -1/2 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1/2 \\ 1/2 & 0 \end{pmatrix} & \begin{pmatrix} 1 & -3/5 \\ -3/5 & 1/5 \end{pmatrix} \begin{pmatrix} -3/5 & 1/5 \\ 1/5 & 1/5 \end{pmatrix} \\
 \begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} & \begin{pmatrix} -3/5 & 1/5 \\ 1/5 & 1/5 \end{pmatrix} \begin{pmatrix} 1/5 & 1/5 \\ 1/5 & -3/5 \end{pmatrix}
 \end{array}$$

Gröebner Basis elements

$$n : x_{1111}^2 x_{1221} x_{2121} x_{2212} x_{1122} - x_{1211} x_{2111} x_{1121}^2 x_{1112} x_{2222}$$

$$n : x_{1111}^5 x_{2211} x_{1221} x_{2121} x_{2221} x_{1212} x_{2112} x_{2212} x_{1122} x_{1222} x_{2122} - x_{1211}^3 x_{2111}^3 x_{1121}^3 x_{1112}^3 x_{2222}^3$$

Markov move in $\text{Ker}_z(A, m^{(0)})$

$$\begin{array}{cc}
 n=2(e^+ - e^-) = & n=5(e^+ - e^-) = \\
 \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 5 & -3 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix} \\
 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}
 \end{array}$$

Alternative table (solution)

$$m^{(1)} = m^{(0)} + n = \begin{array}{cc}
 \mathbf{4(a)} & \mathbf{4(b)} \\
 \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\
 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}
 \end{array}$$

Complete Independence Models

- k-dimensional table
- fixed 1-D marginals (MSS)
- admit square-free Markov Basis (D 2004)
- non-integer extreme points (C 2003)

$$\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$$

- so, CI models non-network

Example 5 2x2x2 Complete Independence Model

$$m^{(0)} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

Kernel element

$$e^+ - e^- = \begin{pmatrix} -1 & 1/2 \\ 1/2 & 0 \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$$

Gröebner Basis element

$$n: x_{121}x_{211}x_{112} - x_{111}^2x_{222}$$

(Minimal) Markov move in $\text{Ker}_Z(A, m^{(0)})$

$$n = 2(e^+ - e^-) = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Alternative table (solution)

$$m^{(1)} = m^{(0)} + n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

CI model admits square-free Markov Basis, but
 n is a minimal Markov move from $m^{(0)}$

$$m^{(0)} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

$$m^{(0)} + n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$n^{(1)} = \begin{pmatrix} -1 & 0 \\ +1 & 0 \end{pmatrix} \begin{pmatrix} 0 & +1 \\ 0 & -1 \end{pmatrix} \quad n^{(2)} = \begin{pmatrix} -1 & +1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} +1 & -1 \\ 0 & 0 \end{pmatrix}$$

$$m^{(1)} = m^{(0)} + n^{(1)} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$$(m^{(0)} + n^{(1)}) + n^{(2)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = m^{(0)} + n$$

Explanation

- LP changed from $m^{(0)}$ to $m^{(1)}$
- $n^{(2)}$ not a feasible move from $m^{(0)}$

THE LP METHOD

Move $n=(n^+ - n^-) \in Ker_z(A, m^{(0)}); n^+, n^- \in N^d$

Kernel element $x=(x^+ - x^-) \in Ker(A, m^{(0)}); 0 \leq x^+, x^- \leq 1$

$$Supp(n^+) = \{i : n_i^+ > 0\}, \quad Val(n^+) = \sum_i n_i^+$$

$$Supp(n^-) = \{i : n_i^- > 0\}, \quad Val(n^-) = \sum_i n_i^-$$

$n = n^+ - n^-$ is *irredundant* iff $Supp(n^+) \cap Supp(n^-) = \emptyset$

$x = x^+ - x^-$ is *irredundant* iff $Supp(x^+) \cap Supp(x^-) = \emptyset$

$Supp(e), Supp(x)$ well-defined for irredundant e, x

$Val(e), Val(x)$ well-defined for irredundant e, x

Move $n = n^+ - n^-$ *minimal* iff 1) cannot be expressed as the sum of two moves of smaller support and 2) has minimum valence among moves with same support

Moves (points) with support = $Supp(n)$ (= $Supp(x)$) lie on the line from 0 thru n (thru x)

Point $(x^+, x^-) \in LP(A, m^{(0)})$ is *irreducible* iff it cannot be expressed as the sum of two points with strictly smaller support

$(x^+, x^-) \in LP(A, m^{(0)})$ is a *convex combination* of extreme points of $LP(A, m^{(0)})$

Extreme points have minimal support, hence

Extreme points are irreducible:

If $e = x + y$ with $Supp(x), Supp(y) \subset Supp(e)$, then
 $Supp(x) = Supp(e)$

Extreme points are irreducible points of maximum valence among points with the same support

Irreducible points lie on extreme rays, viz.,

$(x^+, x^-) \in LP(A, m^{(0)})$ is irreducible iff it is a multiple of an extreme point $e = (e^+, e^-) \in LP(A, m^{(0)})$

Irreducible points on the boundary $\leftarrow \rightarrow$ Extreme points

Theorem: Potential minimal move $n \leftarrow \rightarrow$ Extreme point e

Proof: (\leftarrow) Given extreme point $e=(e^+, e^-)$ of $LP(A, m^{(0)})$, continue on the ray from 0 thru e to the first integer point (n^+, n^-) on or outside the polytope. This point defines a potential minimal move $n(e)=n^+ - n^- \in Ker_z(A)$, called the *e-canonical (potential) minimal move*. The $m^{(0)}$ -feasibility constraint determines if $n(e)=n^+ - n^-$ is feasible from $m^{(0)}$ or not, viz., is or is not a **move** from $m^{(0)}$.

(\rightarrow) Given minimal move $n=n^+ - n^-$, let $x^{(0)} \in F$ denote the point where the ray from irredundant integer point (n^+, n^-) thru 0 meets the boundary of $LP(A, m^{(0)})$. So $(n^+, n^-)=tx^{(0)}$, $t \in \mathbb{Z}$, and $x^{(0)}$ is irreducible. Let e denote an extreme point of $LP(A, m^{(0)})$ minimizing objective function $d(x) = -\sum_{j \in Supp(n)} (x_j^+ + x_j^-)$

As $\min\{d(x)\} \leq d(n)$, then $Supp(e) \subset Supp(n)$. Then either $n = n(e)$ or $n = n(e) + (n - n(e))$ is a decomposition of n into moves with smaller support, a contradiction. **Q.E.D.**

THE PROCEDURE

Random cost function

$$c(x) = \sum_j c_j (x_j^+ - x_j^-)$$

$c_j \in \{-1, 0, 1\}$ uniform random selection

Metropolis step

same as Diaconis-Sturmfels (1998)

Iterative step

- return \mathbf{A} to its starting position
- conform the feasibility constraint to $m^{(0)}$
- generate a uniformly random cost function
- run the LP to obtain extreme point e
- select the proposal move $n =$ first integer point along the ray from 0 thru e
- apply Metropolis decision criterion
- move or stay

Theorem: The Markov chain is irreducible.

Proof: Any move is a sum of minimal moves.

Extreme points correspond to potential minimal Markov moves from $m^{(0)}$.

Every extreme point e has nonzero probability of being generated because to each extreme point e there corresponds an *e-canonical cost function* $c^{(e)}$ that minimizes at only this extreme:

$$c^{(e)}(x^+ - x^-) = \sum_{j \in \text{Supp}(e^-)} (x_j^+ - x_j^-) + \sum_{j \in \text{Supp}(e^+)} (x_j^- - x_j^+)$$

Q.E.D.

Theorem: The Markov chain is reversible.

Proof: Assume that the procedure generates uniformly random objective function $c(x^+ - x^-)$ at current solution $m^{(0)}$, and that, from the common starting presentation of \mathbf{A} , $LP(A, m^{(0)})$ produces extreme point $e = (e^+, e^-)$ minimizing $c(x^+ - x^-)$. Let $n = n^+ - n^-$ denote the e -canonical move from $m^{(0)}$ and let $m^{(1)} = m^{(0)} + n$.

The procedure will generate objective function $-c(x^+ - x^-)$ from $m^{(1)}$ with equal probability. If zero-restrictions imposed by the $m^{(0)}$ - and $m^{(1)}$ -feasibility constraints are identical, then, as each begins with the same presentation of \mathbf{A} , the linear programs $\min\{c(x^+ - x^-) : LP(A, m^{(0)})\}$ and $\min\{-c(x^+ - x^-) : LP(A, m^{(1)})\} = \min\{c(x^- - x^+) : LP(A, m^{(1)})\}$ will follow identical sequences of steps to reach an optimal solution—the only, and key, difference being reversal of the roles of the x^+ - and x^- -variables. That is, subject to identical zero-restrictions, the procedure is equally likely to produce move $-n$ from $m^{(1)}$ as it was to produce move n from $m^{(0)}$. So, under the condition of identical zero restrictions, the proposal density is **symmetric** and the Markov chain is reversible.

When the zero-restrictions are not identical, $-n$ continues to correspond to an optimizing extreme point of $\min\{-c(x^+ - x^-) : LP(A, m^{(1)})\}$, but differences in presentation of the $m^{(0)}$ - and $m^{(1)}$ -LPs may force the simplex to select a different extreme point as the optimal

solution. However, the LP $\min\{c^*(x^+ - x^-) : LP(A, m^{(1)})\}$

with
$$c_j^* = -c_j \quad j \in \text{Supp}(e)$$

$$c_j^* = 0 \quad \text{otherwise}$$

is equiprobable with $c(x)$ and optimizes uniquely at (e^-, e^+) , the extreme point corresponding to $-n$.

Thus, the proposal density is symmetric in all cases and the Markov chain is reversible. **Q.E.D.**

Corollary: The procedure avoids infeasible solutions.

The procedure respects all global and local zero-restrictions, and e is a feasible solution to the local LP

It is possible that potential minimal move $n=te$ is infeasible

Example

Potential minimal move 4(b) is infeasible for table 4(a)

$m^{(0)}$		$n=5e$	
$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 5 & -3 \\ -3 & 1 \end{pmatrix}$	$\begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix}$
$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$	$\begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}$

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